M. E. STRUCTURAL ENGINEERING

II SEMESTER

CZSEPE 21 FINITE ELEMENT METHOD IN STRUCTURAL ENGINEERING

COURSE MATERIAL

[UNIT IV & V]

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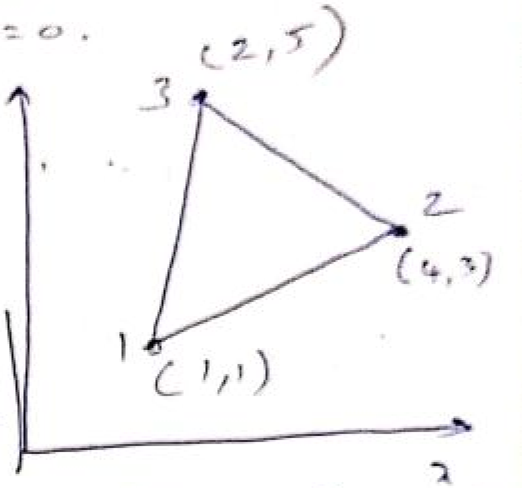
Ex. 4.10

For the triangular element shown in figure, derive the strain displacement matrix [B] and stiffness matrix [k] using

i) Explicit Integration for Plane Stress Problem

ii) Using isoparametric formulation with one Point quadrature rule.

Take $E = 2\times10^3\ kN/cm^2$ & $\mu = 0$.



P25 i) $[B] = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \end{bmatrix}$

P24 $A = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 2 & 5 \end{vmatrix}$

$= \frac{1}{2}\left[1(20-6) - 1(5-3) + 1(2-4)\right] = \frac{1}{2}\left[14-2-2\right]$

$\boxed{A = 5}$

$b_1 = y_2 - y_3 = 3-5 = -2$

$b_2 = y_3 - y_1 = 5-1 = 4$

$b_3 = y_1 - y_2 = 1-3 = -2$

$a_1 = x_3 - x_2 = 2-4 = -2$

$a_2 = x_1 - x_3 = 1-2 = -1$

$a_3 = x_2 - x_1 = 4-1 = 3$

$$[B] = \frac{1}{2\times5}\begin{bmatrix} -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 \\ -2 & -1 & 3 & -2 & 4 & -2 \end{bmatrix}$$

Assuming constant thickness h, $K = h\iint [B]^T[C][B]\,dx\,dy$

$= Ah\,[B]^T[C][B]$

Plane stress Problem,

$$[C] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} = 2\times10^3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$\underset{6\times6}{[k]} = Ah\times\frac{1}{10}\underset{6\times3}{\begin{bmatrix} -2 & 0 & -2 \\ 4 & 0 & -1 \\ -2 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -1 & 4 \\ 0 & 3 & -2 \end{bmatrix}} 2\times10^3 \underset{3\times3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}} \frac{1}{10}\underset{3\times6}{\begin{bmatrix} -2 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 \\ -2 & -1 & 3 & -2 & -4 \end{bmatrix}}$$

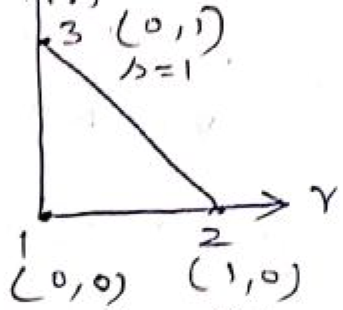
4-8'

$$= 100h \begin{bmatrix} -2 & 0 & -1 \\ 4 & 0 & -1/2 \\ -2 & 0 & 3/2 \\ 0 & -2 & -1 \\ 0 & -1 & 2 \\ 0 & 3 & -1 \end{bmatrix}_{6\times3} \begin{bmatrix} -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 \\ -2 & -1 & 3 & -2 & 4 & -2 \end{bmatrix}_{3\times6}$$

$$[k] = 100h \begin{bmatrix} 6 & -7 & 1 & 2 & -4 & 2 \\ -7 & 33/2 & -\frac{19}{2} & 1 & -2 & 1 \\ 1 & -\frac{19}{2} & \frac{17}{2} & -3 & 6 & -3 \\ 2 & 1 & -3 & 6 & -2 & -4 \\ -4 & -2 & 6 & -2 & 9 & -7 \\ 2 & 1 & -3 & -4 & -7 & 11 \end{bmatrix}_{6\times6}$$

(ii) Isoparametric formulation and numerical Integration.

(i) $[N] = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix}$



$u = \alpha_1 + \alpha_2 r + \alpha_3 s$ — ①

$\{u\} = \begin{bmatrix} 1 & r & s \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$

P26

$\{u\} = [\phi]\{\alpha\}$

(ii) From figure, Natural coordinates of nodes in eqn ①

@node 1, r=s=0, $u_1 = \alpha_1$

@node 2, r=1, s=0, $u_2 = \alpha_1 + \alpha_2$

@node 3, r=0, s=1, $u_3 = \alpha_1 + \alpha_3$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

displacement transformation matrix

generalized coordinates.

Nodal degree of freedom

vector of nodal displacement

$\{d_u\} = [A]\{\alpha\}$

$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & [illegible] \end{bmatrix}$ ✓

$\{\alpha\} = [A]^{-1}\{d\}$ in u, $\{u\} = [\phi][A]^{-1}\{d\}$
$= [N]\{d\}$

$[A]^{-1} = \dfrac{C^T}{|A|}$

$$C = \begin{bmatrix} +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$|A| = 1(1-0) - 0 + 0 = 1$

$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ ✓

$$[N] = [\phi][A]^{-1}$$

$$= [\underset{\longrightarrow}{1 \ r \ s}] \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1\downarrow & 0 & 1 \end{bmatrix}$$

New Shape Function ✓

$$[N] = \begin{bmatrix} (1-r-s) & r & s \end{bmatrix} = [N_1 \ N_2 \ N_3]$$

$N_1 = 1-r-s$ $N_2 = r$ $N_3 = s$

$$\{x\} = [N]\{x_n\}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{Bmatrix}$$

$$\therefore \quad x = N_1x_1 + N_2x_2 + N_3x_3$$

$$y = N_1y_1 + N_2y_2 + N_3y_3$$

Jacobian.

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}$$

$$\frac{\partial x}{\partial r} = \frac{\partial N_1}{\partial r}x_1 + \frac{\partial N_2}{\partial r}x_2 + \frac{\partial N_3}{\partial r}x_3$$

$$[J]_{2\times2} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \\ \frac{\partial N_1}{\partial s} & \frac{\partial N_2}{\partial s} & \frac{\partial N_3}{\partial s} \end{bmatrix}_{2\times3} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}_{3\times2}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$|J| = 3\times4 - 1\times2 = 10$$

$$[J]^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} J_{11}^{*} & J_{12}^{*} \\ J_{21}^{*} & J_{22}^{*} \end{bmatrix}$$

$$\begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \begin{Bmatrix} \partial/\partial r \\ \partial/\partial s \end{Bmatrix}$$ (P45) → eq ①

$$\underset{4\times1}{\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{Bmatrix}} = \underset{4\times4}{\begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ J_{21}^* & J_{22}^* & 0 & 0 \\ 0 & 0 & J_{11}^* & J_{12}^* \\ 0 & 0 & J_{21}^* & J_{22}^* \end{bmatrix}} \underset{4\times1}{\begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{Bmatrix}}$$

$*$ is represented by the inverse.

$$\frac{\partial u}{\partial x} = J_{11}^* \frac{\partial u}{\partial r} + J_{12}^* \frac{\partial u}{\partial s}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} \\ \frac{\partial v}{\partial s} \end{Bmatrix}$$

$$\{u\} = [N]\{d\}$$

$$\{d\}^T = [\, u_1 \;\; u_2 \;\; u_3 \;\; v_1 \;\; v_2 \;\; v_3 \,]$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$\frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r} u_1 + \frac{\partial N_2}{\partial r} u_2 + \frac{\partial N_3}{\partial r} u_3$$

$$\therefore \underset{3\times1}{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}} = \underset{3\times4}{\begin{bmatrix} J_{11}^* & J_{12}^* & 0 & 0 \\ 0 & 0 & J_{21}^* & J_{22}^* \\ J_{21}^* & J_{22}^* & J_{11}^* & J_{12}^* \end{bmatrix}} \underset{4\times6}{\begin{bmatrix} \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} & 0 & 0 & 0 \\ \frac{\partial N_1}{\partial s} & \frac{\partial N_2}{\partial s} & \frac{\partial N_3}{\partial s} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial r} \\ 0 & 0 & 0 & \frac{\partial N_1}{\partial s} & \frac{\partial N_2}{\partial s} & \frac{\partial N_3}{\partial s} \end{bmatrix}} \underset{6\times1}{\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}}$$

vector of strains

strain displacement matrix

(P34) $$\{\varepsilon\} = [B]\{d\}$$

vector of nodal displacement

$$\underset{3\times 6}{[B]} = \frac{1}{10}\underset{3\times 4}{\begin{bmatrix} 4 & -2 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ -1 & 3 & 4 & -2 \end{bmatrix}}\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}_{4\times 6}$$

$$= \frac{1}{10}\begin{bmatrix} -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 \\ -2 & -1 & 3 & -2 & 4 & -2 \end{bmatrix}$$

Using one Point quadrature rule,

$$[k] = \frac{1}{2} \sum h\, w_i [B]_i^T [C] [B]_i |J|_i$$

$$r = s = 1/3$$

$$w_i = 1.0$$

$$[k] = \frac{1}{2} \times h \times 1\, [B]^T [C] [B]\, |J|$$

$$= \frac{1}{\cancel{2}} h \times \underset{|J|}{10} \times \cancel{\frac{1}{10}} \times \cancel{\frac{1}{10}} \times \overset{E}{(\cancel{2} \times 10^{3})} \begin{bmatrix} -2 & 0 & -2 \\ 4 & 0 & -1 \\ -2 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -1 & 4 \\ 0 & 3 & -2 \end{bmatrix}_{6\times 3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}_{3\times 3}$$

$$\times \begin{bmatrix} -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 \\ -2 & -1 & 3 & -2 & 4 & -2 \end{bmatrix}_{3\times 6}$$

plane stress

$$[C] = \frac{E}{1-\mu^2} \times \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{pmatrix}$$

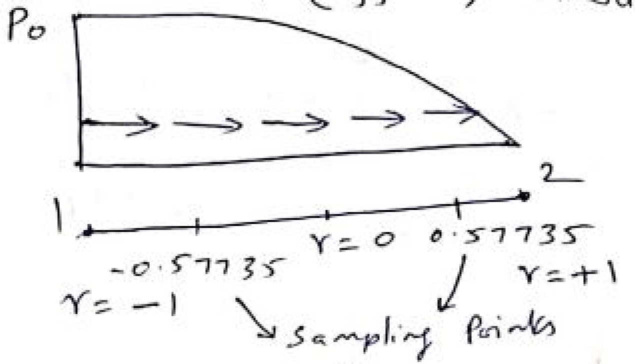
$$\boxed{\mu = 0}$$

$$= 100h \begin{bmatrix} -2 & 0 & -1 \\ 4 & 0 & -1/2 \\ -2 & 0 & 3/2 \\ 0 & -2 & -1 \\ 0 & -1 & 2 \\ 0 & 3 & -1 \end{bmatrix}_{6\times 3} \begin{bmatrix} -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 3 \\ -2 & -1 & 3 & -2 & 4 & -2 \end{bmatrix}_{3\times 6}$$

$$= 100h \begin{bmatrix} 6 & -7 & 1 & 2 & -4 & 2 \\ -7 & 33/2 & -\frac{19}{2} & 1 & -2 & 1 \\ 1 & -19/2 & 17/2 & -3 & 6 & -3 \\ 2 & 1 & -3 & 6 & -2 & -4 \\ -4 & -2 & 6 & -2 & 9 & -7 \\ 2 & 1 & -3 & -4 & -7 & 11 \end{bmatrix}_{6\times 6}$$

Both [B] & [k] matrices derive by explicity integration and numerical integration are found to be the same.

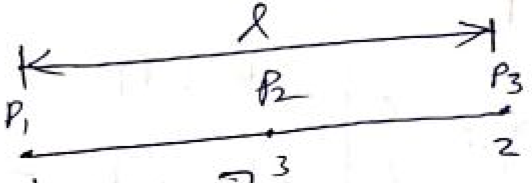
P144 An axial element with 2 nodes is subjected to a varying load $P = P_0\left[1-\left(\frac{x}{l}\right)^2\right]$. Compute the nodal load vector {Q} by numerical integregation. (50)



$P_1 = P_0$
$(x=0)$

$P_2 = 0$
$(x=l)$

$P_3 = \frac{3}{4}P_0$
$x = \frac{l}{2}$



$$[N] = \left[-\frac{1}{2}r(1-r) \quad \frac{1}{2}r(1+r) \quad (1-r^2)\right]$$

$$P = [N]\{P\} = [N_1 \quad N_2 \quad N_3]\begin{Bmatrix}P_1\\P_2\\P_3\end{Bmatrix}$$

$$= \left[-\frac{1}{2}r(1-r) \quad \frac{1}{2}r(1+r) \quad (1-r^2)\right]\begin{Bmatrix}P_1\\P_2\\P_2\end{Bmatrix}$$

$$\{Q\} = \int [N^s]^T \{\overset{\text{surface load}}{p}\}\, dx$$

$$[N^s] = \left[\frac{1-r}{2} \quad \frac{1+r}{2}\right]$$ for two noded line element

$$\underset{2\times1}{\{Q\}} = \underset{2\times1}{\begin{Bmatrix}\frac{1-r}{2}\\ \frac{1+r}{2}\end{Bmatrix}} \underset{1\times3}{\left[-\frac{1}{2}r(1-r) \quad \frac{1}{2}r(1+r) \quad (1-r^2)\right]} \underset{3\times1}{\begin{Bmatrix}P_1\\P_2\\P_2\end{Bmatrix}} \frac{l}{2}dr$$ (P40

$$\{Q\} = \frac{l}{8}\int \underset{2\times3}{\begin{bmatrix}-r(1-r)^2 & r(1-r^2) & 2(1-r)(1-r^2)\\ -r(1-r^2) & r(1+r)^2 & 2(1+r)(1-r^2)\end{bmatrix}} \underset{3\times1}{\begin{Bmatrix}P_1\\P_2\\P_3\end{Bmatrix}} dr$$

Using two points Gauss quadrature, Numerical integration $W_1 = W_2 = 1.0$

$r_1 = -0.57735$, $r_2 = +0.57735$

$$\int f(r) = \sum_{i=1}^{2} w_i f(r_i)$$

$$\{Q\} = \frac{l}{8}\begin{bmatrix} -(-0.57735)(1-(-0.57735))^2 P_1 + (-0.57735)(1-(-0.57735^2))P_2 \quad 2(1-(-0.57735))(1-(-0.57735^2))P_3 \\ -(-0.57735)(1-(-0.57735)^2)P_1 \quad -0.57735(1-0.57735)^2 P_2 \quad 2(1-0.57735)(1-(-0.57735)^2)P_3 \end{bmatrix}$$

$$+ \frac{l}{8}\begin{bmatrix} -0.57735(1-0.57735)^2 P_1 \cdot 0.57735(1-0.57735^2)P_2 \quad 2(1+0.57735)(1-0.57735^2)P_3 \\ -0.57735(1-0.57735^2)P_1 \quad 0.57735(1+0.57735)^2 P_2 \quad 2(1+0.57735)(1-0.57735^2)P_3 \end{bmatrix}$$

$$\{Q\} = \frac{l}{8}\begin{bmatrix} 1.436P_1 & \overset{0}{\cancel{-0.385P_2}} & 2.103P_3 \\ 0.385P_1 & \overset{0}{\cancel{-0.103P_2}} & 0.564P_3 \end{bmatrix}$$

$$+ \frac{l}{8}\begin{bmatrix} -0.103P_1 & \overset{0}{\cancel{0.385P_2}} & 0.564P_3 \\ -0.385P_1 & \overset{0}{\cancel{1.436P_2}} & 2.103P_3 \end{bmatrix}$$

$P_2 = 0$
$P_1 = P_0$
$P_3 = \frac{3}{4}P_0$

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{l}{8}\begin{Bmatrix} 1.436P_0 + \overset{1.57725}{2.103 \times \frac{3}{4}P_0} - 0.103P_0 + \overset{0.423}{0.564 \times \frac{3}{4}P_0} \\ 0.385P_0 + 0.564 \times \frac{3}{4}P_0 - 0.385P_0 + 2.103 \times \frac{3}{4}P_0 \end{Bmatrix}$$

$$= \frac{l}{8}\begin{Bmatrix} 3.333P_0 \\ 2.0P_0 \end{Bmatrix} = \begin{Bmatrix} 0.417P_0 l \\ 0.25P_0 l \end{Bmatrix}$$

Total load $= Q_1 + Q_2 = P_0 l(0.417 + 0.25) = 0.667P_0 l$

Actual load $= \frac{2}{3}P_0 l = 0.667P_0 l$

Two dimensional Truss Element

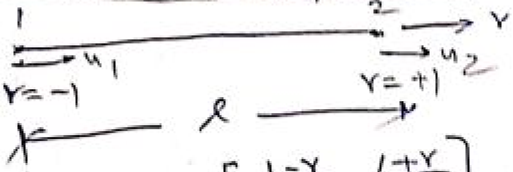
(P21) Natural coordinates

(P31) Shape function :

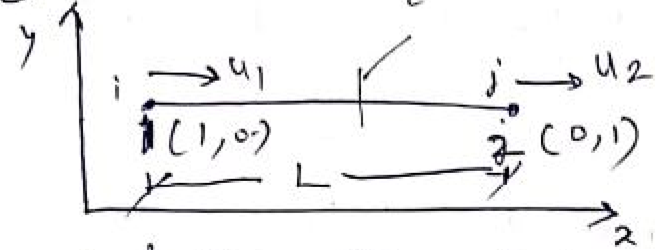
(P36b) [K]

(P35) [B]

$\boxed{r \text{ coordinate system}}$



$$N = \left[\frac{1-r}{2} \quad \frac{1+r}{2}\right]$$



$\boxed{L_1 L_2 \text{ System}}$

2 Dl - Truss Element

Figure shows a 2 Dl truss element which is Parallel to x axis and has nodes i and j. Since it is an axial force resisting member the displacement along x axis will be u and its variation can be expressed as

$$u = L_1 u_1 + L_2 u_2 \qquad — ①$$

$$u = \alpha_1 L_1 + \alpha_2 L_2$$

$$= [L_1 \; L_2] \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

(P26)

$$\{u\} = [\phi] \{\alpha\}$$

$$\therefore [\phi] = [L_1 \; L_2]$$

In matrix form

$$\{d\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

In ①

(Node 1) $L_1 = 1$, $L_2 = 0$

$$u_1 = \alpha_1 (1) + \alpha_2 (0) = \alpha_1$$

(Node 2) $L_1 = 0$, $L_2 = 1$

$$u_2 = \alpha (0) + \alpha_2 (1) = \alpha_2$$

$$[N] = [\phi][A]^{-1}$$

unit matrix $[A] = [A]^{-1}$

$$= [L_1 \; L_2]_{1\times2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$$

$$[N] = [L_1 \; L_2]_{1\times2}$$

(P26)

$$\{d\} = [A]\{\alpha\}$$

$$\{\alpha\} = [A]^{-1}\{d\}$$

$$\therefore \{u\} = [\phi][A]^{-1}\{d\}$$

$$= [N]\{d\}$$

$$\{u\} = [N]\{d\} = \overset{N_1 = L_1 \quad N_2 = L_2}{[L_1 \; L_2]} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u = L_1 u_1 + L_2 u_2$$

P21

Strain variation:

$$\epsilon = \frac{\partial u}{\partial x} = \frac{1}{l}\left(\frac{\partial u}{\partial L_2} - \frac{\partial u}{\partial L_1}\right)$$

$$= \frac{1}{l}(u_2 - u_1)$$

$$= \frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix}$$

P34

$$\{\epsilon\} = [B]\{d\}$$

Strain-displacement matrix [B]

$$\boxed{[B] = \frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix}}$$

⇒ P35

Axial stress in element $\sigma = E\epsilon$

$$= [C]\epsilon$$

$$\therefore [C] = E$$

Stiffness matrix for truss element, which is prismatic and having an area of cross section, A, can be computed as follows:

$$[K_m] = \iiint [B]^T[C][B]\,dv$$

$$= A\int [B]^T[C][B]\,dl$$

$$[K_m] = \frac{AE}{l^2}\int \begin{Bmatrix}-1\\ 1\end{Bmatrix}_{(2\times 1)} \begin{bmatrix}-1 & 1\end{bmatrix}_{(1\times 2)} dl$$

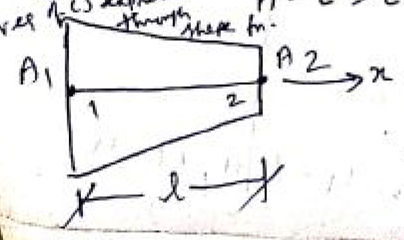
$$= \frac{AE}{l^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}_{2\times 2} \boxed{\int dl}$$

Stiffness matrix of element/member

$$\boxed{[K_m] = \frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}$$

P36b

in local member axis system i.e. corresponding to degrees of freedom at the nodes i and j in local member axes system.



$$A = [N]\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} = [L_1 L_2]\begin{Bmatrix}A_1\\ A_2\end{Bmatrix}$$

$$A = [L_1, L_2]\begin{Bmatrix}A_1\\ A_2\end{Bmatrix}$$ L_1 & L_2 are natural coordinates.

P21b

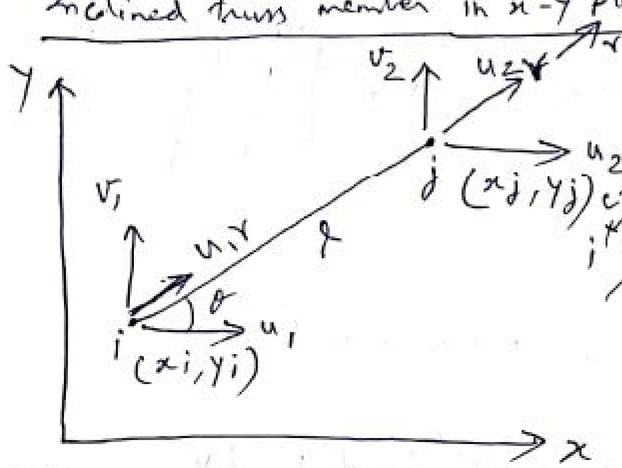
$$[K_M] = A\int [B]^T[C][B]\,dl = E\int \frac{1}{l^2}\begin{Bmatrix}-1\\ 1\end{Bmatrix}_{2\times 1}\begin{bmatrix}-1 & 1\end{bmatrix}_{1\times 2}\underbrace{[L_1 L_2]}_{}\overset{(l-x)/l \quad x/l}{}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix} dl$$

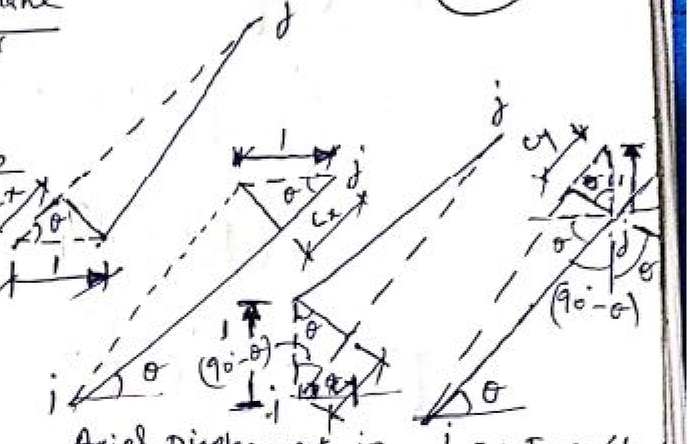
$$= \frac{E}{l^2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\int (A_1 L_1 + A_2 L_2)\,dl = \int_0^l (A_1 - A_1 x/l + A_2 x/l)\,dl$$

$$= A_1 l - \frac{A_1 l^2}{2l} + \frac{A_2 l^2}{2l} = \frac{A_1 l}{2} + \frac{A_2 l}{2} = \left(\frac{A_1 + A_2}{2}\right)l$$

$$= \frac{E(A_1 + A_2)}{2l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \;//$$

Inclined truss member in x-y plane





Axial Displacements in 2D Truss Element

Stiffness matrix for this element with respect to displacement in global x-y axes will be derived.

Axial displacements ~~[illegible]~~ produced in the element due to unit displacements given at nodes i and j are shown in Fig.

Direction cosines : $\frac{C_x}{1} = \cos\theta = \frac{x_j - x_i}{l}$

$$\frac{C_y}{1} = \sin\theta = \frac{y_j - y_i}{l}$$

& $l = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$

where x_i, y_i ; x_j, y_j are the coordinates of nodes i and j.

Displacement variation in the member direction i.e r direction is given by :

$$u_r = L_1 u_{1r} + L_2 u_{2r}$$

u_{1r} and v_{2r} are the displacements of nodes i and j in r direction.

(P51)

$\{u\} = [N]\{d\}$ $\{u_1 \atop u_2\}$
$= [L_1 L_2]$ $\{u_1 \atop u_2\}$

$u = L_1 u_1 + L_2 u_2$

Axial nodal displacements can be expressed in terms of global displacements u_1, v_1, u_2 and v_2 at nodes i and j.

$$u_{1r} = u_1\cos\theta + v_1\sin\theta = C_x u_1 + C_y v_1$$

$$u_{2r} = u_2\cos\theta + v_2\sin\theta = C_x u_2 + C_y v_2$$

Axial strain ϵ_r along member axis r of element is

(P51b) $\epsilon_r = \frac{1}{l}(u_{2r} - u_{1r})$

$$\epsilon_x = \frac{1}{l}\left[c_x u_2 + c_y v_2 - c_x u_1 - c_y v_1\right]$$

$$= \frac{1}{l}\left[-c_x \;\; -c_y \;\; c_x \;\; c_y\right]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= [B]\{d\}$$

$$[B] = \frac{1}{l}\left[-c_x \;\; -c_y \;\; c_x \;\; c_y\right]$$

$$[k_m] = \iiint [B]^T \overset{E}{[C]} [B]\, dv$$

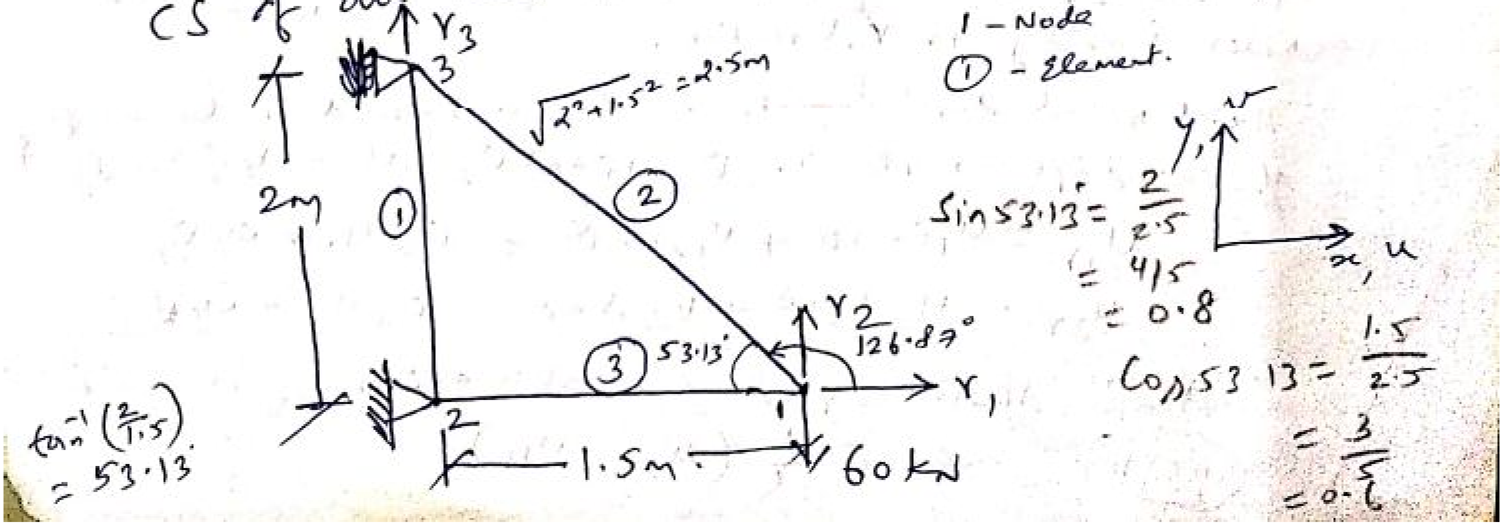
$$= \iiint \frac{E}{l^2}\begin{Bmatrix} -c_x \\ -c_y \\ c_x \\ c_y \end{Bmatrix}_{4\times1}\left[-c_x \;\; -c_y \;\; c_x \;\; c_y\right]_{1\times4} dV$$

k in global axis system

$$= \frac{AE}{l^2}\int\begin{bmatrix} c_x^2 & c_x c_y & -c_x^2 & -c_x c_y \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 \\ -c_x^2 & -c_x c_y & c_x^2 & c_x c_y \\ -c_x c_y & -c_y^2 & c_x c_y & c_y^2 \end{bmatrix}_{4\times4} dl$$

$$= \frac{AE}{l}\left[\begin{array}{cc|cc|c} \downarrow u_1 & \downarrow v_1 & \downarrow u_2 & \downarrow v_2 & \\ c_x^2 & c_x c_y & -c_x^2 & -c_x c_y & u_1 \\ c_x c_y & c_y^2 & -c_x c_y & -c_y^2 & v_1 \\ \hline -c_x^2 & -c_x c_y & c_x^2 & c_x c_y & u_2 \\ -c_x c_y & -c_y^2 & c_x c_y & c_y2 & v_2 \end{array}\right]$$

Find out the forces in the inclined member of truss shown in figure. Assume the area of CS of all members are same. $E = 2\times10^5\ \mathrm{N/mm^2}$.



$$\tan^{-1}\left(\frac{2}{1.5}\right) = 53.13°$$

$$\sin 53.13° = \frac{2}{2.5} = 4/5 = 0.8$$

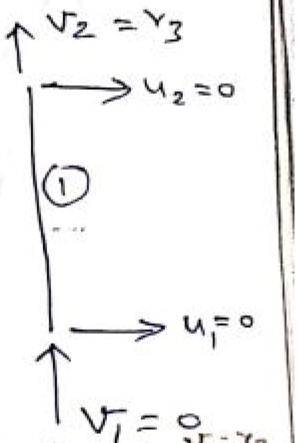
$$\cos 53.13 = \frac{1.5}{2.5} = \frac{3}{5} = 0.6$$

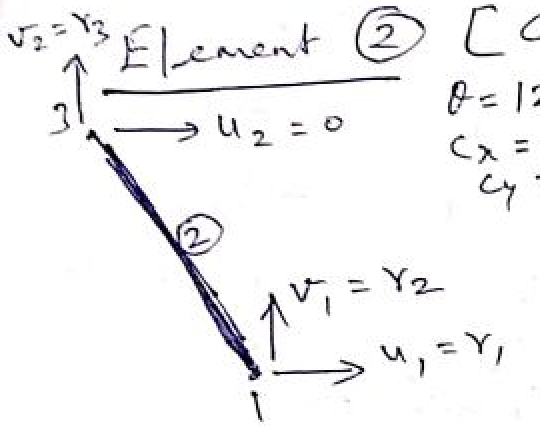
Element ① [Connectivity 2-3]

(53)

$\theta = 90^\circ$ $\quad C_x = 0$ ($\cos 90^\circ = 0$) $\quad C_y = 1$ ($\sin 90^\circ = 1$) $\quad l = 2m$.

$$[k]_1 = \frac{AE}{2.0}\begin{array}{ccccc} u_1=0 & v_1=0 & u_2=0 & v_2=r_3 & \\ 0 & 0 & 0 & 0 & u_1=0 \\ -0 & 1 & 0 & -1 & v_1=0 \\ 0 & 0 & 0 & 0 & v_2=0 \\ 0 & -1 & 0 & 1 & v_2=r_3 \\ 0 & 0 & 0 & 3 & \end{array}$$





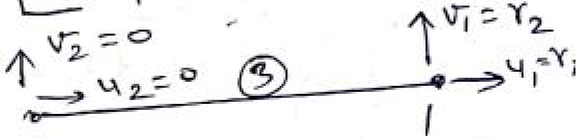
Element ② [Connectivity 1-3]

$\theta = 126.9^\circ$ $\quad l = 2.5m$

$C_x = -0.6$

$C_y = 0.8$

$$[k]_2 = \frac{AE}{2.5}\begin{array}{ccccc} u_1=r_1 & v_1=r_2 & u_2=0 & v_2=r_3 & \\ 0.36 & -0.48 & -0.36 & 0.48 & 1 \\ -0.48 & 0.64 & 0.48 & -0.64 & 2 \\ -0.36 & 0.48 & 0.36 & -0.48 & 0 \\ 0.48 & -0.64 & -0.48 & 0.64 & 3 \\ 1 & 2 & 0 & 3 & \end{array}$$



Element ③ [Connectivity 1-2]

$\theta = 180^\circ$ $\quad C_x = \cos 180^\circ = -1$ $\quad \sin 180^\circ = C_y = 0$ $\quad l = 1.5m$

$$[k]_3 = \frac{AE}{1.5}\begin{array}{ccccc} & 2 & 0 & 0 & \\ 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & & & & \end{array}$$

$$[k] = AE\begin{array}{cccc} 1 & 2 & 3 & \\ \frac{0.36}{2.5} = 0.144 \quad 1/1.5 = 0.667 & -0.192 & 0.192 & 1 \\ -0.192 & 0.256 & -0.256 & 2 \\ 0.192 & -0.256 & \frac{1}{2} = 0.5 \quad 0.256 & 3 \end{array}$$

$$= AE\begin{bmatrix} +0.811 & -0.192 & 0.192 \\ -0.192 & 0.256 & -0.256 \\ 0.192 & -0.256 & 0.756 \end{bmatrix}$$

Action

Displacement

$$\boxed{A = SD}$$

Stiffness

$$[k]_{YxY}\{r\}_{Yx1} = \{Q\}_{Yx1}$$

$$AE\begin{bmatrix} 0.811 & -0.192 & 0.192 \\ -0.192 & 0.256 & -0.256 \\ 0.192 & -0.256 & 0.756 \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -60 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \frac{1}{AE}\begin{bmatrix} 0.811 & -0.192 & 0.192 \\ -0.192 & 0.256 & -0.256 \\ 0.192 & -0.256 & 0.756 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ -60 \\ 0 \end{Bmatrix}$$

A^{-1} ... see Annexure

$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} -\frac{67.464}{AE} \\ -405/AE \\ -\frac{120}{AE} \end{Bmatrix}$$

$\therefore$ $r_1 = -\frac{67.466}{AE}$

$r_2 = -\frac{405.0}{AE}$

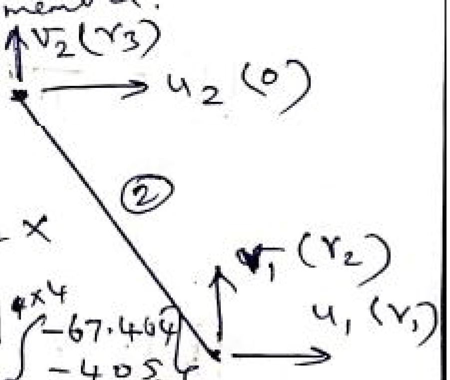
$r_3 = -\frac{120}{AE}$

The end forces in the inclined member 2 along global direction can be given as

$$\{S\}_2 = [\overset{\text{stiffness matrix for the member}}{k}]_2 \{d\}_2$$

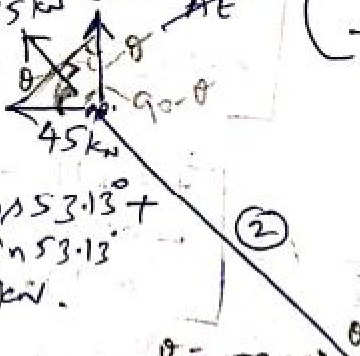
← displacement matrix for the member.

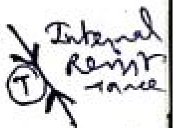
$$\{d\}_2 = \frac{1}{AE}\begin{Bmatrix} -67.464 \\ -405 \\ -120 \end{Bmatrix}\begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$



$$\{S\}_2 = \frac{AE}{2.5}\begin{bmatrix} 0.36 & -0.48 & -0.36 & 0.48 \\ -0.48 & 0.64 & 0.48 & -0.64 \\ -0.36 & 0.48 & 0.36 & -0.48 \\ 0.48 & -0.64 & -0.48 & 0.64 \end{bmatrix}_{4\times4} \times \frac{1}{AE}\begin{Bmatrix} -67.464 \\ -405 \\ 0 \\ -120 \end{Bmatrix}_{4\times1}$$

$$\{S\}_{2\,(4\times1)} = \begin{Bmatrix} 45 \\ -60 \\ -45 \\ 60 \end{Bmatrix}\begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$





SOM (away from jt)

Force in member ② = 45 Cos 53.13° + 60 sin 53.13° = 75kN.

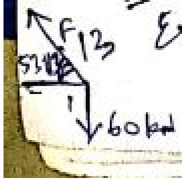
Member End force

SOM: method of joints

@ node 1

$\Sigma v = 0 = -60 + F_{13} \sin 53.13$

$F_{13} = \frac{60}{0.8} = 75\text{kN}$ (T)



θ = 53.13°

θ = 53.13° 45 kN

$\sin 53.33 = \frac{x}{60}$ $(90° - θ)$

$x = 0.8 \times 60 = 48\text{kN}$

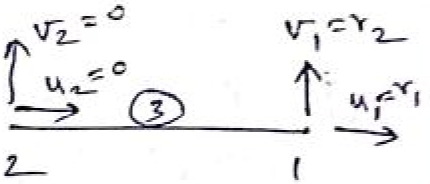
$\cos 53.13 = \frac{x}{45}$

$x = 45 \times 0.6$

$= 27\text{kN}$

End forces in horizontal member ③

$$\{S\}_3 = [k]_3 \{d\}_3$$

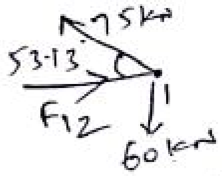


$$\{d\}_3 = \frac{1}{AE}\left\{\begin{array}{rl} -67.466 & u_1 \\ -405 & v_1 \\ 0 & u_2 \\ 0 & v_2 \end{array}\right\}$$

$$\{S\}_3 = \frac{\cancel{AE}}{1.5}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\left\{\begin{array}{r} -67.466 \\ -405 \\ 0 \\ 0 \end{array}\right\}\frac{1}{\cancel{AE}}$$

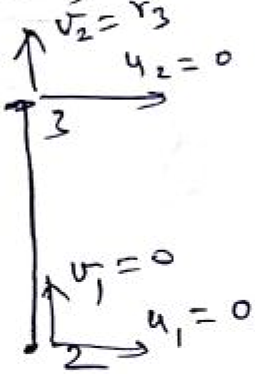
$$= \left\{\begin{array}{rl} -45.0 & u_1 \\ 0 & v_1 \\ +45.0 & u_2 \\ 0 & v_2 \end{array}\right\}$$

Som



$\Sigma H = 0$, $\quad F_{12} - 75 \cos 53.13° = 0$

$F_{12} = 75 \times 0.6 = \underline{\underline{45.0 \text{ kN}}}$ (C)



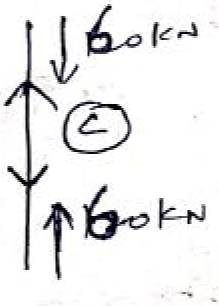
End forces in vertical member ①

$$\{S\}_1 = [k]_1 \{d\}_1$$

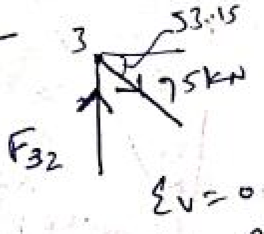
$$\{d\}_1 = \frac{1}{AE}\left\{\begin{array}{rl} 0 & u_1 \\ 0 & v_1 \\ 0 & u_2 \\ -120 & v_2 \end{array}\right\}$$

$$\{S\}_1 = \frac{\cancel{AE}}{2.0}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\frac{1}{\cancel{AE}}\left\{\begin{array}{rl} 0 & u_1 \\ 0 & v_1 \\ 0 & u_2 \\ -120 & v_2 \end{array}\right\}$$

$$= \left\{\begin{array}{rl} 0 & u_1 \\ 60 & v_1 \\ 0 & u_2 \\ -60 & v_2 \end{array}\right\}$$



Som

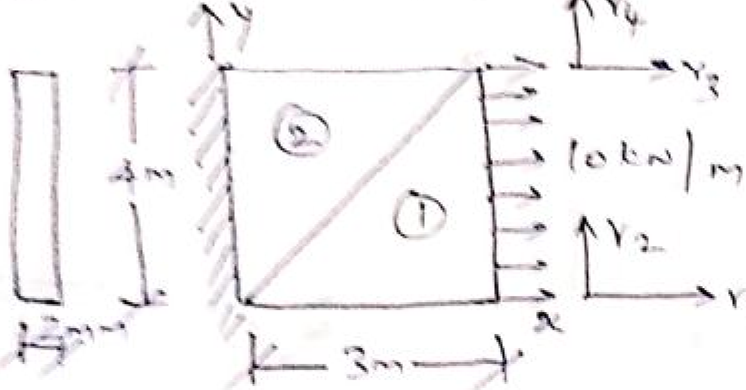


$\Sigma V = 0 = F_{32} - 75 \sin 53.15°$

$F_{32} = 75 \times 0.8 = \underline{\underline{60.0 \text{ kN}}}$ ✓ (C)

Plane Stress and Plane Strain Analysis.

① A rectangular plate shown in figure is subjected to a UDL. Analyse the plate using two CST elements, to find σ_x, σ_y and τ_{xy}. Take $E = 2.1\times10^{11}\ N/m^2$ and $\mu = 0.3$.



$E = 2.1\times10^{8}\ kN/m^2$

r_1, r_2, r_3, r_4 are nodal dof. = Unknown joint displacement.

For plane stress conditions:

P13b

$$[C] = \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

$$= \frac{2.1\times10^{8}}{1-0.3^2}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & \frac{1-0.3}{2} \end{bmatrix}$$

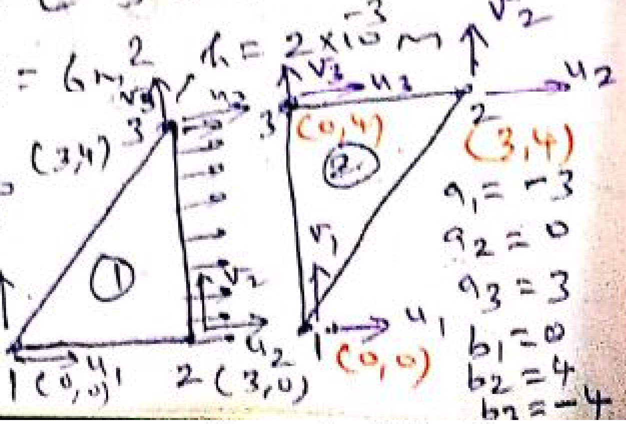
$$[C] = 2.3077\times10^{8}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

Element Stiffness matrix [K];

P26

$$[K] = Ah\,[B]^T[C][B]$$

$$A = \frac{1}{2}\times4\times3 = 6m^2 \qquad h = 2\times10^{-3}\ m$$



$b_1 = y_2 - y_3 = -4$

$b_2 = y_3 - y_1 = 4$

$b_3 = y_1 - y_2 = 0$

P23b

$a_1 = x_3 - x_2 = 3-3 = 0$

$a_2 = x_1 - x_3 = -3$

$a_3 = x_2 - x_1 = 3$

$a_1 = -3$

$a_2 = 0$

$a_3 = 3$

$b_1 = 0$

$b_2 = 4$

$b_3 = -4$

$$[B] = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & b_1 & b_2 & b_3 \end{bmatrix}$$

$$[B]_1 = \frac{1}{12}\begin{bmatrix} -4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & -3 & 3 & -4 & 4 & 0 \end{bmatrix}$$

$$[B]_2 = \frac{1}{12}\begin{bmatrix} 0 & 4 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ -3 & 0 & 3 & 0 & 4 & -4 \end{bmatrix}$$

$$[K]_1 = 6\times2\times10^{-3}\times\frac{1}{12}\times\frac{1}{12}\times2.3077\times10^{8}$$

$$\underset{6\times3}{\begin{bmatrix} -4 & 0 & 0 \\ 4 & 0 & -3 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \\ 0 & -3 & 4 \\ 0 & 3 & 0 \end{bmatrix}}\underset{3\times3}{\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}\underset{3\times6}{\begin{bmatrix} -4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & -3 & 3 & -4 & 4 & 0 \end{bmatrix}}$$

$$= 1.923\times10^{4}\underset{6\times3}{\begin{bmatrix} -4 & -1.2 & 0 \\ 4 & 1.2 & -1.05 \\ 0 & 0 & 1.05 \\ 0 & 0 & -1.4 \\ -0.9 & -3 & 1.4 \\ 0.9 & 3 & 0 \end{bmatrix}}\underset{3\times6}{\begin{bmatrix} -4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & -3 & 3 & -4 & 4 & 0 \end{bmatrix}}$$

$$= 1.923\times10^{4}\;\begin{array}{cccccccc}
 & r1 & r3 & 0 & r2 & r4 & \Leftarrow g.dof & \\
u_1 & u_2 & u_3 & v_1 & v_2 & v_3 & \Leftarrow e.dof & \\
16 & -16 & 0 & 0 & 3.6 & -3.6 & u_1\downarrow & 0\downarrow \\
-16 & 19.15 & -3.15 & 4.2 & -7.8 & 3.6 & u_2 & r1 \\
0 & -3.15 & 3.15 & -4.2 & 4.2 & 0 & u_3 & r_3 \\
0 & 4.2 & -4.2 & 5.6 & -5.6 & 0 & v_1 & 0 \\
3.6 & -7.8 & 4.2 & -5.6 & 14.6 & -9 & v_2 & r2 \\
3.6 & 3.6 & 0 & 0 & -9 & 9 & v_3 & r_4
\end{array}\;6\times6$$

$$[k]_2 = 1.923\times10^{4}\begin{bmatrix} 0 & 0 & -3 \\ 4 & 0 & 0 \\ -4 & 0 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \\ 0 & 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}[B]_2$$

$$= 1.923\times10^{3}\begin{bmatrix} 0 & 0 & -1.05 \\ 4 & 1.2 & 0 \\ -4 & -1.2 & 1.05 \\ -0.9 & -3 & 0 \\ 0 & 0 & 1.4 \\ 0.9 & 3 & -1.4 \end{bmatrix}[B]_2$$

$$[k]_2 = 1.923\times10^4 \begin{bmatrix} 0 & 0 & -1.05 \\ 4 & 1.2 & 0 \\ -4 & -1.2 & 1.05 \\ -0.9 & -3 & 0 \\ 0 & 0 & 1.4 \\ 0.9 & 3 & -1.4 \end{bmatrix} \begin{bmatrix} 0 & 4 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ -3 & 0 & 3 & 0 & 4 & -4 \end{bmatrix} \leftarrow$$

$$[k]_2 = 1.923\times10^4 \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc|cc} 0 & r_3 & 0 & 0 & r_4 & 0 & \text{g.dof} \downarrow & \\ u_1 & u_2 & u_3 & v_1 & v_2 & v_3 & \text{e.dof} \downarrow & \\ 3.15 & 0 & -3.15 & 0 & -4.2 & 4.2 & u_1 & 0 \\ 0 & \bigcirc\!\!\!\!16 & -16 & -3.6 & \bigcirc\!\!\!\!0 & 3.6 & u_2 & r_3 \\ -3.15 & -16 & 19.15 & 3.6 & 4.2 & -7.8 & u_3 & 0 \\ 0 & -3.6 & 3.6 & 9 & 0 & -9 & v_1 & 0 \\ -4.2 & \bigcirc\!\!\!\!0 & 4.2 & 0 & \bigcirc\!\!\!\!5.6 & -5.6 & v_2 & r_4 \\ 4.2 & 3.6 & -7.8 & -9 & -5.6 & 14.6 & v_3 & 0 \end{array}$$

Element load vectors:

$$\{Q_1\} = \frac{P\ell}{2}\begin{Bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{array}{c} \\ \end{array}\begin{Bmatrix} 0 \\ 20 \\ 20 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\begin{array}{cc} \text{e.dof} & \text{g.dof} \\ u_1 & 0 \\ u_2 & r_1 \\ u_3 & r_3 \\ v_1 & 0 \\ v_2 & r_2 \\ v_3 & r_4 \end{array}$$

$$\{Q_2\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}\begin{array}{cc} u_1 & 0 \\ u_2 & 0 \\ u_3 & 0 \\ v_1 & 0 \\ v_2 & 0 \\ v_3 & 0 \end{array}$$

Assembling of [k] matrix

$$[k] = 1.923\times10^4 \begin{array}{cccc|c} 1 & 2' & 3 & 4 & \text{g.dof} \downarrow \\ 19.15 & -7.8 & -3.15 & 3.6 & 1 \\ -7.8 & 14.6 & 4.2 & -9.0 & 2 \\ -3.15 & 4.2 & \substack{3.15 \\ 16} & \substack{0 \\ 0} & 3 \\ 3.6 & -9 & \substack{0 \\ 0} & \substack{9.0 \\ 5.6} & 4 \end{array}$$

$$\{Q\} = \begin{Bmatrix} 20 \\ 0 \\ 20 \\ 0 \end{Bmatrix} \begin{array}{c} \text{g.dof} \\ \rightarrow r_1 \\ r_2 \\ \rightarrow r_3 \\ r_4 \end{array}$$

Assembling like [k]

$$\{Q\} = \{Q_1\} + \{Q_2\} = \begin{Bmatrix} 20 \\ 0 \\ 20 \\ 0 \end{Bmatrix}\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array}$$

Solution of displacement

$$[k]\{r\} = \{Q\}$$

$$1.923\times10^{4}\begin{bmatrix} 19.15 & -7.8 & -3.15 & 3.6 \\ -7.8 & 14.6 & 4.2 & -9.0 \\ -3.15 & 4.2 & 19.15 & 0 \\ 3.6 & -9.0 & 0 & 14.6 \end{bmatrix}\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 20 \\ 0 \end{Bmatrix}$$

Solving by Gaussian Elimination

$r_1 = 7.251\times10^{-5}$; $r_2 = 1.521\times10^{-5}$

$r_3 = 6.288\times10^{-5}$; $r_4 = -0.881\times10^{-5}$

Element Stresses

Element ①:

$$\{\sigma_1\} = [C][B_1]\{d_1\} \quad ; \quad \{d_1\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ r_1 \\ r_3 \\ 0 \\ r_2 \\ r_4 \end{Bmatrix}$$

$$\{\sigma_1\} = 2.3077\times10^{8}\underset{3\times3}{\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}\frac{1}{12}\underset{3\times6}{\begin{bmatrix} -4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 3 \\ 0 & -3 & 3 & -4 & 4 & 0 \end{bmatrix}} \times \begin{Bmatrix} 0 \\ 7.251\times10^{-5} \\ 6.288\times10^{-5} \\ 0 \\ 1.521\times10^{-5} \\ -0.881\times10^{-5} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 1.923\times10^{7}\underset{3\times6}{\begin{bmatrix} -4 & +4 & 0 & 0 & 0 & -0.9 & 0.9 \\ -1.2 & 1.2 & 0 & 0 & -3 & 3 \\ 0 & -1.05 & 1.05 & -1.4 & 1.4 & 0 \end{bmatrix}}\begin{Bmatrix} 0 \\ 7.251\times10^{-5} \\ 6.288\times10^{-5} \\ 0 \\ 1.521\times10^{-5} \\ -0.881\times10^{-5} \end{Bmatrix}$$

$$\{\sigma_1\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 5{,}577.47 \\ 287.53 \\ 215.04 \end{Bmatrix}$$

Element ② $\{\sigma_2\} = [C][B_2]\{d_2\} = 2.3077\times10^{8}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}\frac{1}{12}$

$$\times\begin{bmatrix} 0 & +4 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 3 \\ -3 & 0 & 3 & 0 & +4 & -4 \end{bmatrix}\{d_2\} = 1.923\times10^{7}\begin{bmatrix} 0 & 4 & -4 & -0.9 & 0 & 0.9 \\ 0 & 1.2 & -1.2 & -3 & 0 & 3 \\ -1.05 & 0 & 1.05 & 0 & 1.4 & -1.4 \end{bmatrix}$$

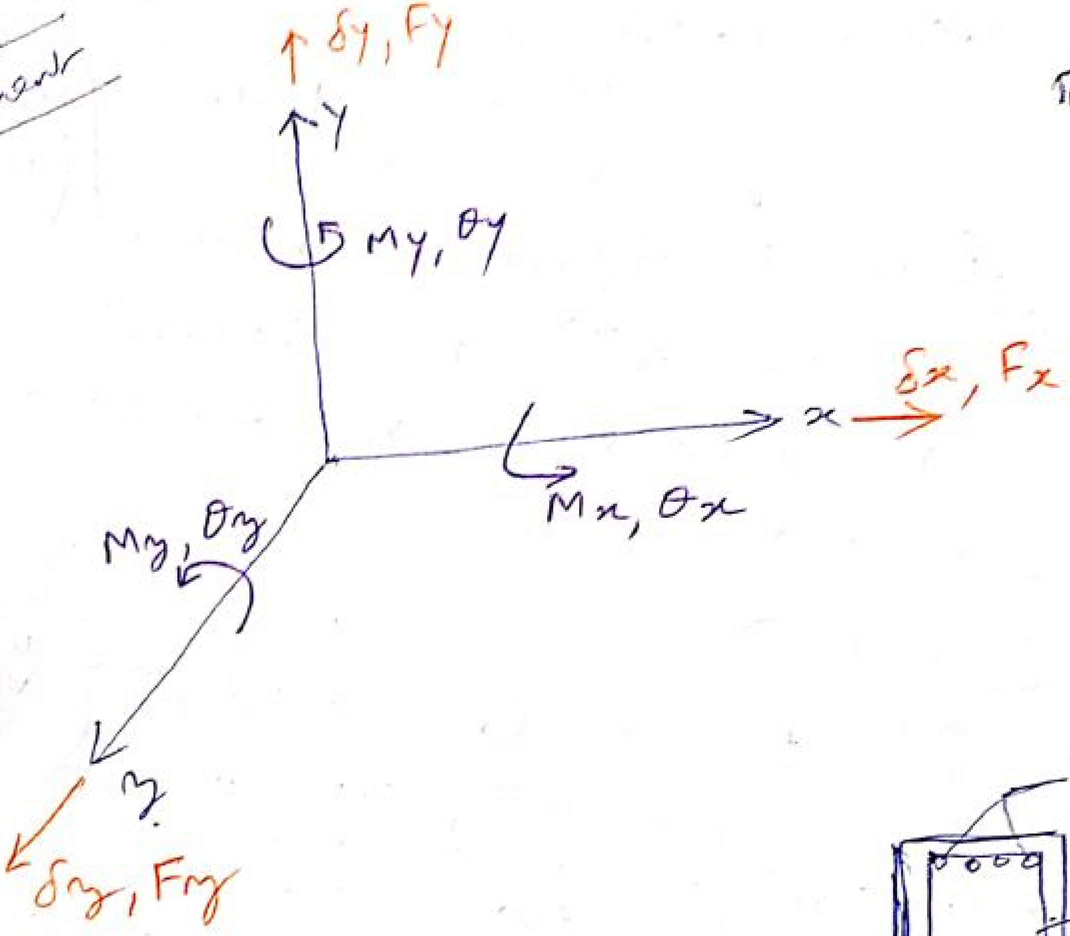
$$= 1.923\times10^{7}\begin{bmatrix} 0 & 4 & -4 & -0.9 & 0 & 0.9 \\ 0 & 1.2 & -1.2 & -3 & 0 & 0.3 \\ -1.05 & 0 & 1.05 & 0 & 1.4 & -1.4 \end{bmatrix}\begin{Bmatrix} 0 \\ 6.288\times10^{-5} \\ 0 \\ 0 \\ -0.881\times10^{-5} \\ 0 \end{Bmatrix} \leftarrow \begin{Bmatrix} u_1 = 0 \\ u_2 = r_3 \\ u_3 = 0 \\ v_1 = 0 \\ v_2 = r_4 \\ v_3 = 0 \end{Bmatrix}$$

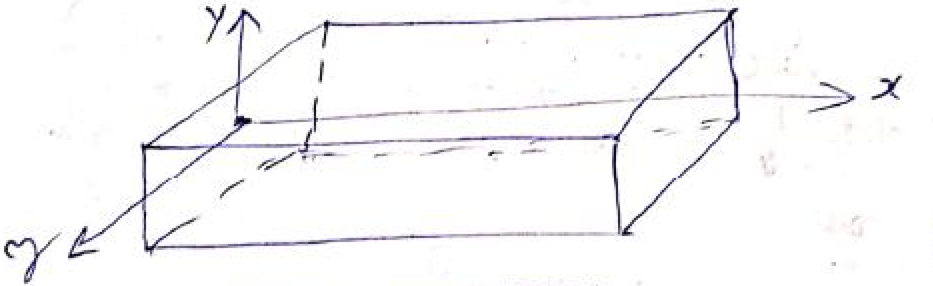
$$= \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 4836.73 \\ 1451.02 \\ -237.18 \end{Bmatrix}$$

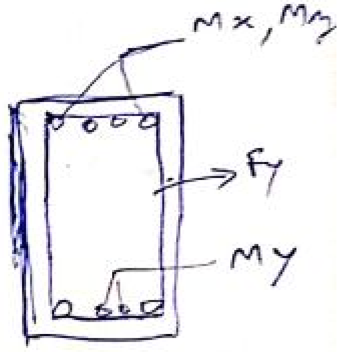
// Check: $\sigma = \frac{P}{A} = \frac{10}{2\times10^{-3}\times1} = 5000\ \text{kN/m}^2$

Beam Element

23/3/20







Shear check: F_z

① Mx, Fy, Mz

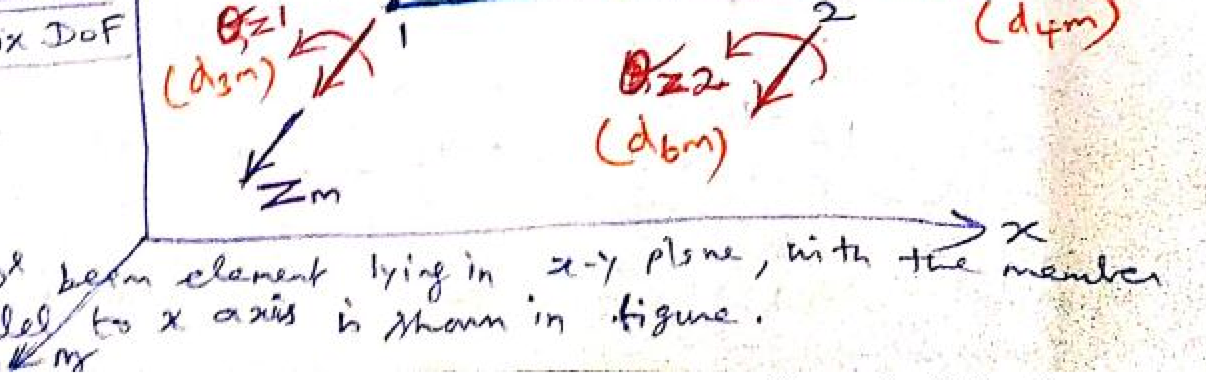
② My, Fz, Fy

Local member axis system (xm, ym, zm) is ||l to global axes (x, y, z)

2 D Beam Element:

(P214 CSK)

Two Dimensional beam element with six DoF

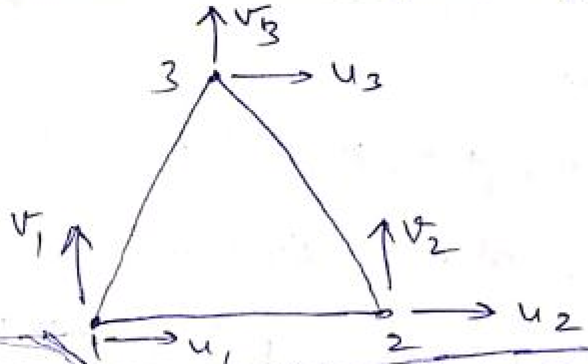


A 2D beam element lying in x-y plane, with the member Parallel to x axis is shown in figure.

K matrix

57

K matrix 2×2



6×6

$$\{d\} = \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \end{Bmatrix} = \{d_M\} = \begin{Bmatrix} d_{1m} \\ d_{2m} \\ d_{3m} \\ d_{4m} \\ d_{5m} \\ d_{6m} \end{Bmatrix} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_{z1} \\ u_2 \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

Global DoF — Element DoF / member DoF

DoF referred to the member axes x_m, y_m, z_m and global axes x, y & z are same.

$u_1 \rightarrow d_1$
$u_2 \rightarrow d_4$

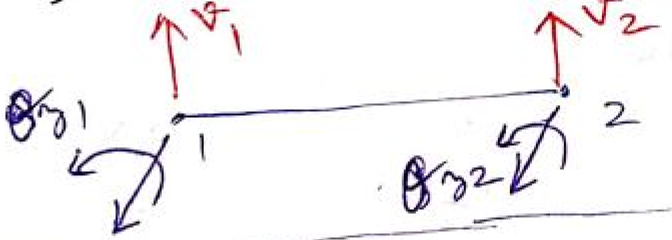
where u_1, v_1 and u_2, v_2 are displacements along x and y axes and θ_{z1} & θ_{z2} are rotations about z axis at nodes 1 and 2.

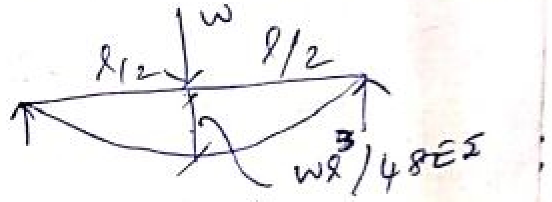
Stiffness coefficients due to axial displacement u_1 & u_2 is

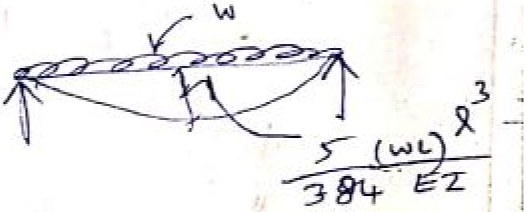
$$K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

They do not influence the response of the member due to displacements v_1, θ_{z1}, v_2 & θ_{z2}.

2 Dl, Beam with 4 DoF







$$\theta = \frac{dy}{dx}$$

$$\theta_z = \frac{dv}{dx}$$

y : Displacement

θ : rotation.

Scanned with CamScanner

The variation of v will be cubic and can be expressed in generalised coordinates as

$$v = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

Natural coordinate system,

$$v = \alpha_1 L_1^3 + \alpha_2 L_2^3 + \alpha_3 L_1^2 L_2 + \alpha_4 L_1 L_2^2 \quad -①$$

P26

$$v = [N]\{d\}$$

$$= [N_1 \quad N_2 \quad N_3 \quad N4] \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

P40 P21

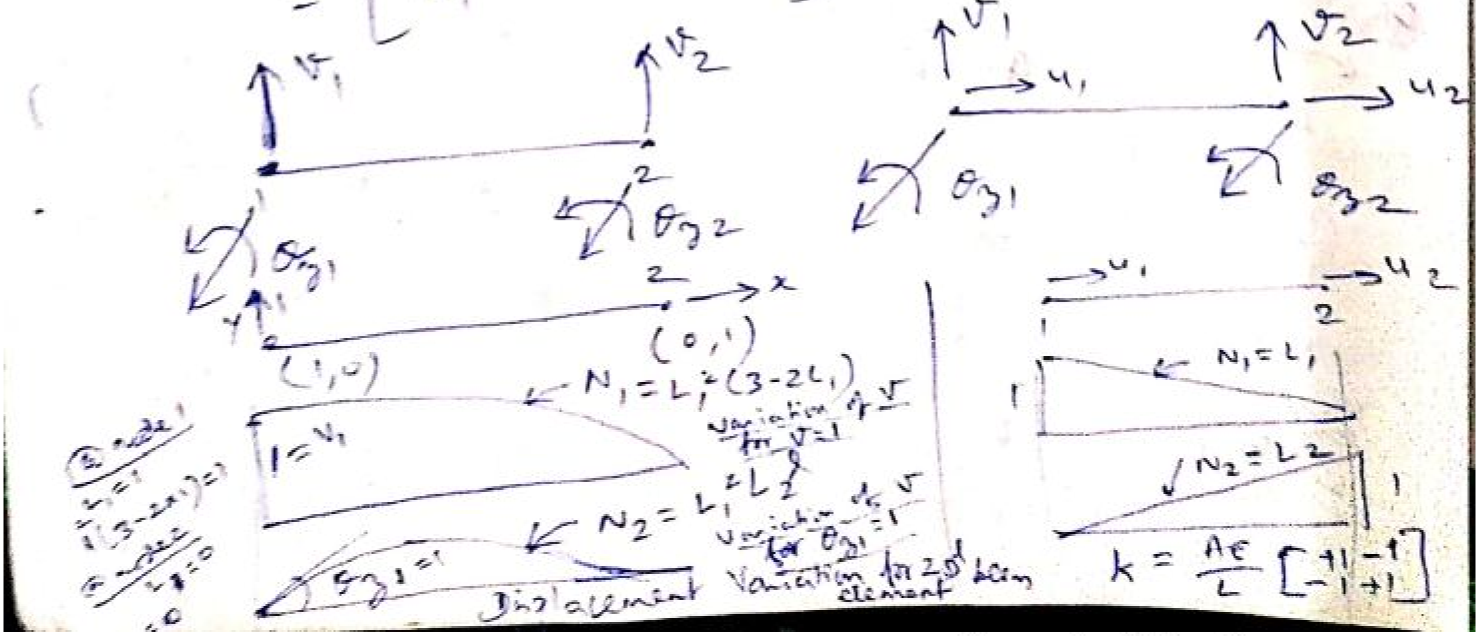
$$\theta_z = \frac{dv}{dx} = \frac{1}{l}\left(\frac{d}{dL_2} - \frac{d}{dL_1}\right) v$$

$$\theta_z = \frac{1}{l}\begin{bmatrix} 3\alpha_2 L_2^2 + \alpha_3 L_1^2 + 2\alpha_4 L_1 L_2 \\ -3\alpha_1 L_1^2 - 2\alpha_3 L_1 L_2 - \alpha_4 L_2^2 \end{bmatrix} \quad -②$$

XX

$$[N] = \left[L_1^2(3-2L_1) \quad L_1^2 L_2 l \quad L_2^2(3-2L_2) \quad -L_1 L_2^2 l \right]$$

$$= [N_1 \quad N_2 \quad N_3 \quad N_4]$$



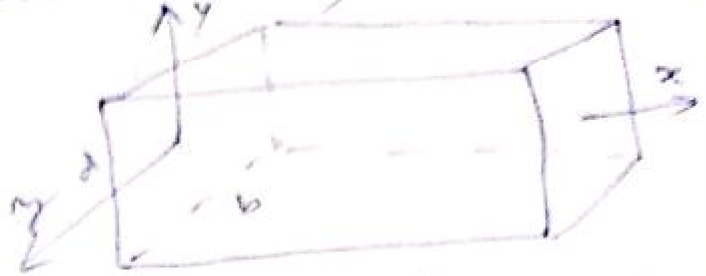
Displacement variation for 2D beam element

$$k = \frac{AE}{L}\begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

⇒ displacement (v) indicates the rotation $\left(\frac{dv}{dx}\right)$

⇒ stiffness matrix is always symmetric

⇒ Shape of stiffness matrix is 6×6, each node is 12



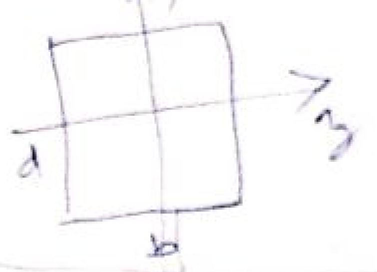
$$[C] = [E I_z]$$

$$I_z = \frac{1}{12} b d^3$$

$$M = E I_z \frac{d^2 v}{dx^2}$$

I_x → Torsional moment

I_z → Flexural "



(xx) $N_i (i = 1, 2, 3, 4)$ can be evaluated more directly by noting that each term N_i gives variation of v for unit value of d_i while all the other displacement components are held zero.

(N₁) , $d_1 = v_1 = 1$, $d_2, d_3, d_4 \Rightarrow 0$

At node i, $v_1 = 1$, $\theta_{z_1} = 0$, at node j, $v_2 = 0$, $\theta_{z_2} = 0$

Eqn ① @ node 1 $L_1 = 1, L_2 = 0$: $v_1 = 1 = \alpha_1 (1)^3 + 0 + 0 + 0 \Rightarrow \boxed{\alpha_1 = 1}$

@ node 2 $L_1 = 0, L_2 = 1$: $v_2 = 0 = 0 + \alpha_2 (1)^3 + 0 + 0 \Rightarrow \boxed{\alpha_2 = 0}$

Eqn ② @ node 1 $L_1 = 1, L_2 = 0$: $\theta_{z_1} = 0 = \frac{1}{l}\left[0 + \alpha_3 (1)^2 + 0 - 3\alpha_1 (1)^2 - 0 - 0\right] = \alpha_3 - 3\overset{1}{\alpha_1}$

$\therefore \boxed{\alpha_3 = 3}$

@ node 2, $L_1 = 0, L_2 = 1$: $\theta_{z_2} = 0 = \frac{1}{l}\left[3\alpha_2 (1)^2 + 0 + 0 - 0 - 0 - \alpha_4 (1)^2\right] = 3\overset{0}{\alpha_2} - \alpha_4$

$\boxed{\alpha_4 = 0}$

In ① $\therefore v = L_1^3 + 3 L_1^2 L_2 = L_1^3 + 3L_1^2 (1 - L_1) = L_1^3 + 3L_1^2 - 3L_1^3 = 3L_1^2 - 2L_1^3$

$= L_1^2 (3 - 2L_1) \quad \therefore \boxed{N_1 = L_1^2 (3 - 2L_1)}$

(N₂) $d_2 = \theta_{z_1} = 1$, $d_1, d_3, d_4 \Rightarrow 0$ i.e. $v_1 = 0$, $\theta_{z_1} = 1$, $v_2 = 0$, $\theta_{z_2} = 0$

In ①, node 1 $v_1 = 0 = \alpha_1 (1)^3 + 0 + 0 + 0 \Rightarrow \boxed{\alpha_1 = 0}$

node 2 $v_2 = 0 = 0 + \alpha_2 (1)^3 + 0 + 0 \Rightarrow \boxed{\alpha_2 = 0}$

In ②, $\theta_{z_1} = 1 = \frac{1}{l}\left[0 + \alpha_3 (1)^2 + 0 - 3\overset{0}{\alpha_1} (1)^2 - 0 - 0\right] \Rightarrow \boxed{\alpha_3 = l}$

$\theta_{z_2} = 0 = \frac{1}{l}\left[3\overset{0}{\alpha_2} (1)^2 + 0 + 0 - 0 - 0 - \alpha_4 (1)^2\right] \Rightarrow \boxed{\alpha_4 = 0}$

In ① $\therefore v = l L_1^2 L_2 \qquad \therefore \boxed{N_2 = L_1^2 L_2 l}$

ⓃN₃ $v_1 = 0,\ \theta_{z_1} = 0,\ v_2 = 1,\ \theta_{z_2} = 0$

qn ① $v_1 = 0 = \alpha_1 (1)^3 + 0 + 0 + 0 \Rightarrow \boxed{\alpha_1 = 0}$

$v_2 = 1 = 0 + \alpha_2 (1)^3 + 0 + 0 \Rightarrow \boxed{\alpha_2 = \text{[illegible]}}$

qn ② $\theta_{z_1} = 0 = \frac{1}{l}\left(0 + \alpha_3 (1)^2 + 0 - 3\underset{\nwarrow 0}{\alpha_1}(1)^2 - 0 - 0\right)$

$\boxed{\alpha_3 = 0}$

$\theta_{z_2} = 0 = \frac{1}{l}\left(3\alpha_2 (1)^2 + 0 + 0 - 0 - 0 - \alpha_4 (1)^2\right)$

$\boxed{\alpha_4 = 3}$

qn ① $v = L_2^3 + 3L_1L_2^2 = L_2^3 + 3L_2^2(1 - L_2) = L_2^3 + 3L_2^2 - 3L_2^3$

$= 3L_2^2 - 2L_2^3 = L_2^2(3 - 2L_2)$

$\therefore \boxed{N_3 = L_2^2(3 - 2L_2)}$

(N4) $v_1 = 0,\ \theta_{z_1} = 0,\ V_2 = 0,\ \theta_{z_2} = 1.$

qn ① $v_1 = 0 = \alpha_1 (1)^3 + 0 + 0 + 0 \Rightarrow \boxed{\alpha_1 = 0}$

$v_1 = 0 = 0 + \alpha_2 (1)^3 + 0 + 0 \Rightarrow \boxed{\alpha_2 = 0}$

$V_2 = 0 = 0 + \alpha_3 (1)^2 + 0 - 3\underset{\nwarrow 0}{\alpha_1}(1)^2 - 0 - 0)$

qn ② $\theta_{z_1} = 0 = \frac{1}{l}($

$\boxed{\alpha_3 = 0}$

$\theta_{z_2} = 1 = \frac{1}{l}\left(0 + \overset{\swarrow 0}{\alpha_2}(1)^2 + 0 - 0 - 0 - \alpha_4 (1)^2\right)$

$\boxed{\alpha_4 = -l}$

qn ① $v = -l L_1 L_2^2$ $\therefore \boxed{N_4 = -L_1L_2^2 l}$

$\therefore [N] = \left[L_1^2(3 - 2L_1) \quad L_1^2 L_2 l \quad L_2^2(3 - 2L_2) \quad -L_1L_2^2 l\right]$

Simple theory of bending.

$\epsilon_x = -y \frac{d^2v}{dx^2}$ // $\sigma_x = -E y \frac{d^2v}{dx^2}$

BM $M = \int_{-h/2}^{h/2} \sigma_x\, y\, b\, dy = \int_{-h/2}^{h/2} -E y \frac{d^2v}{dx^2}\, y\, b\, dy$

$= -E I_z \frac{d^2v}{dx^2}$ //

$b\left(\frac{y^3}{3}\right)_{-h/2}^{h/2}$

$\frac{b}{3}\left(\frac{h^3}{8} + \frac{h^3}{8}\right) = \frac{bh^3}{12}$

$= I_z.$

$$\frac{d^2 v}{dx^2} = \frac{d^2}{dx^2}\left[[N]\{d\}\right]$$

P26 (59)

$v = [N]\{d\}$

$$\frac{d}{dx}[N]\{d\} = \frac{1}{l}\begin{bmatrix} 0 & L_1^2 l & 6L_2 - 6L_2^2 & -2L_1L_2 l \\ (-6L_1+6L_1^2) & -2L_1L_2 l & 0 & +L_2^2 l \end{bmatrix}\{d\}$$

$$\frac{d}{dx} = \frac{1}{l}\left(\frac{\partial}{\partial L_2} - \frac{\partial}{\partial L_1}\right)$$

$$\frac{d^2 [N]\{d\}}{dx^2} = \frac{1}{l^2}\begin{bmatrix} 0 & 0 & l(6-12L_2) & -2L_1 l \\ 0 & -2L_1 l & 0 & 2L_2 l \\ 0 & -2L_1 l & 0 & 2L_2 l \\ (6-12L_1) & 2L_2 l & 0 & 0 \end{bmatrix}\{d\}$$

$$= \frac{1}{l^2}\begin{bmatrix} (6-12L_1) & l(2L_2-4L_1) & (6-12L_2) & l(4L_2-2L_1) \end{bmatrix}\{d\}$$

$$\begin{bmatrix} (6-12L_1) & l(2L_2-4L_1) & (6-12L_2) & l(4L_2-2L_1) \end{bmatrix}$$

$$\therefore M = -\frac{EI_y}{l^2}\begin{bmatrix} (6-12L_1) \end{bmatrix} \times \{d\}$$

Stress resultant, Bm (M), similar to the stress

P12 $\{\sigma\} = [C]\{\epsilon\}$

$\{\epsilon\} = [B]\{d\}$ P34

or, $\{\sigma\} = [C]\,[B]\,\{d\}$, the [B] & [C] matrices for beam element are given by

$$[C] = EI_y$$

$$\& \quad [B] = \frac{-1}{l^2}\begin{bmatrix} (6-12L_1) & l(2L_2-4L_1) & (6-12L_2) & l(4L_2-2L_1) \end{bmatrix}$$

Stiffness matrix for beam element is $\boxed{K}$ P36 366

$$= \int [B]^T [C] [B]\, dl$$

$$[K] = [K_m] =$$

$$[K_m] = \frac{EI_y}{l^4}\int_0^l \begin{bmatrix} (6-12L_1) \\ l(2L_2-4L_1) \\ (6-12L_2) \\ l(4L_2-2L_1) \end{bmatrix}_{4\times1} \begin{bmatrix} (6-12L_1) & l(2L_2-4L_1) & (6-12L_2) & l(4L_2-2L_1) \end{bmatrix}_{1\times4} \times dl$$

$$[K_m] = \frac{EI_y}{l^4}\int_0^l \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} dl$$

$a_{11} = (6-12L_1)^2$, $\overset{21}{a_{12}} = \ell(6-12L_1)(2L_2-4L_1)$, $\underset{41}{a_{14}} = \ell(6-12L_1)(4L_2-2L_1)$

$\overset{31}{a_{13}} = (6-12L_1)(6-12L_2)$,

$a_{22} = \ell^2(2L_2-4L_1)^2$, $\overset{32}{a_{23}} = \ell(2L_2-4L_1)(6-12L_2)$

$a_{33} = (6-12L_2)^2$

$\underset{42}{a_{24}} = \ell^2(2L_2-4L_1)(4L_2-2L_1)$, $a_{44} = L^2(4L_2-2L_1)^2$

$\underset{43}{a_{34}} = \ell(6-12L_2)(4L_2-2L_1)$,

$$\int_0^\ell L_1^p L_2^q \, d\ell = \frac{p!\,q!}{(p+q+1)!}\ell$$

$$\int_0^\ell a_{11}\,d\ell = \int_0^\ell (36 + 144L_1^2 - 144L_1)\,d\ell$$

$$= 36\ell + 144 \times \frac{2!}{3!}\ell - 144 \times \frac{1!}{2!}\ell$$

$$= 36\ell + \overset{24}{\cancel{144}} \times \frac{2}{\cancel{6}}\ell - 72\ell = 36\ell - 24\ell = 12\cancel{\ell} \times \frac{EI_z}{\ell^{\cancel{4}3}}$$

$$\boxed{a_{11} = \frac{12EI_z}{\ell^3}}$$

$$a_{12} = \ell\int_0^\ell \left[12L_2 - 24L_1 - 24L_1L_2 + 48L_1^2\right] d\ell$$

$$= \ell\left[12\frac{1}{2}\ell - 24\cdot\frac{1}{2}\ell - 24\cdot\frac{1.1}{3\times2\times1}\ell + 48\frac{2}{3\times2}\ell\right]$$

$$= \ell\left[6\ell - 12\cancel{\ell} - \cancel{4}\ell + 1\cancel{6}\ell = 6\ell\right] = 6\cancel{\ell^2} \times \frac{EI_z}{\ell^{\cancel{4}2}}$$

$$\boxed{a_{12} = \frac{6EI_z}{\ell^2}}$$

$$a_{13} = \int_0^\ell \left[36 - 72L_2 - 72L_1 + 144L_1L_2\right] d\ell$$

$$= 36\ell - \cancel{\overset{36}{\cancel{72}} \cdot \frac{1}{\cancel{2}}\ell - \overset{36}{72}\frac{1}{\cancel{2}}\ell} + \overset{24}{\cancel{144}}\frac{1}{\cancel{6}}\ell$$

$$= -12\cancel{\ell} \times \frac{EI_z}{\ell^{\cancel{4}3}}$$

$$\boxed{a_{13} = \frac{-12EI_z}{\ell^3}}$$

$$a_{14} = l(6-12L_1)(4L_2-2L_1)$$

$$= l(24L_2 - 12L_1 - 48L_1L_2 + 24L_1)$$

$$= l\left[24\cdot\frac{1}{2}l - 12\frac{1}{2}l - 48\cdot\frac{1}{6}l + 24\frac{2}{6}l\right]$$

$$= l\left[12l - 6l - \cancel{8l} + \cancel{8l}\right] = 6\cancel{l^2}\cdot\frac{EI_y}{\cancel{l^4}\,2}$$

$$\boxed{a_{14} = \frac{6EI_y}{l^2}}$$

$$a_{22} = l^2(2L_2-4L_1)^2 = l^2(4L_2^2 + 16L_1^2 - 16L_1L_2)$$

$$= l^2\left(4\cdot\frac{2}{6}l + 16\frac{2}{6}l - 16\frac{1\times1}{6}l\right)$$

$$= l^2\left(\frac{8l}{6} + \frac{32l}{6} - \frac{16l}{6}\right) = l^2\left(\frac{24}{6}l\right)\frac{EI}{l^4}$$

$$\boxed{a_{22} = \frac{4EI_y}{l}}$$

$$a_{23} = l(2L_2-4L_1)(6-12L_2)$$

$$= l\left[12L_2 - 24L_2^2 - 24L_1 + 48L_1L_2\right]$$

$$= l\left[12\cdot\frac{1}{2}l - 24\frac{2}{6}l - 24\frac{1}{2}l + 48\frac{1}{6}l\right]$$

$$= l\left[6l - \cancel{8l} - 12l + \cancel{8l} = -6l\right]\cdot\frac{EI_y}{l^4}$$

$$\boxed{a_{23} = -\frac{6EI_y}{l^2}}$$

$$a_{24} = l^2(2L_2-4L_1)(4L_2-2L_1) = l^2\left(8L_2^2 - 4L_1L_2 - 16L_1L_2 + 8L_1^2\right)$$

$$= l^2\left[8\cdot\frac{2}{6}l - 4\cdot\frac{1}{6}l - 16\cdot\frac{1}{6}l + 8\cdot\frac{2}{6}l\right)$$

$$= l^2\left[\frac{16}{6}l - \frac{4}{6}l - \cancel{\frac{16}{6}}l + \cancel{\frac{16}{6}}l = \frac{12}{6}l = 2l\right]$$

$$= 2l^3\cdot\frac{EI_y}{l^4} = \boxed{\frac{2EI_y}{l} = a_{24}}$$

$$a_{33} = (6-12L_2)^2 = 36 + 144L_2^2 - 144L_2$$

$$= 36l + \overset{48}{\cancel{144}} \cdot \frac{\cancel{2}}{\cancel{6}_{3}} l - \overset{72}{\cancel{144}} \cdot \frac{1}{\cancel{2}} l = 12l \cdot \frac{EI_z}{l^4}$$

$$\boxed{a_{33} = \frac{12EI_z}{l^3}}$$

$$a_{34} = l(6-12L_2)(4L_2-2L_1)$$

$$= l[24L_2 - 12L_1 - 48L_2^2 + 24L_1L_2]$$

$$= l[24 \cdot \frac{1}{2} l - 12 \cdot \frac{1}{2} l - 48 \cdot \frac{2}{6} l + 24 \frac{1}{6} l]$$

$$= l[\cancel{12}l - 6l - \cancel{16}l + \cancel{4}l = -6l] = -6l^2$$

$$\boxed{a_{34} = \frac{-6EI_z}{l^2}}$$

$$a_{44} = l^2(4L_2 - 2L_1)^2 = l^2(16L_2^2 + 4L_1^2 - 16L_1L_2)$$

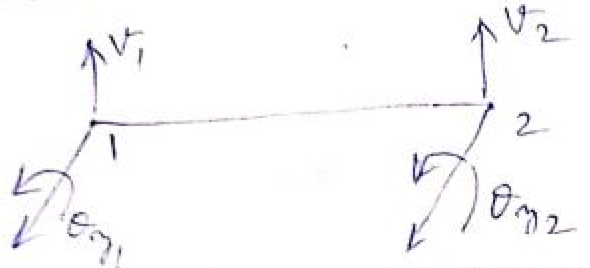
$$= l^2(16 \cdot \frac{2}{6} l + 4 \cdot \frac{2}{6} l - 16 \frac{1}{6} l)$$

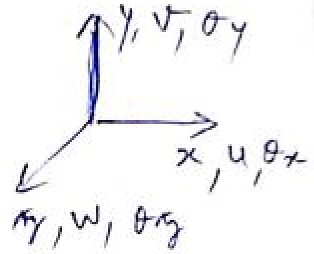
$$= l^2(\frac{32l}{6} + \frac{8l}{6} - \frac{16}{6} l = \frac{\overset{4}{\cancel{24}}}{\cancel{6}} l)$$

$$\boxed{a_{44} = \frac{4EI_z}{l}}$$

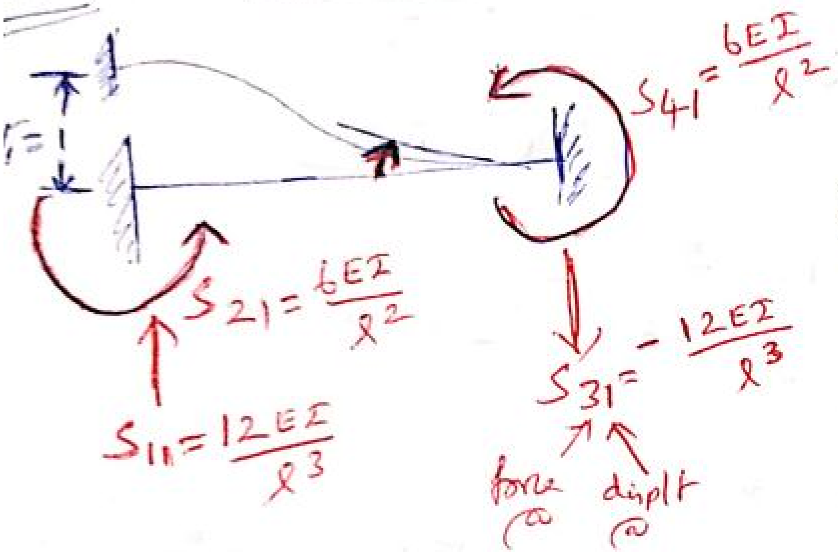
$$[K_m] = \begin{array}{cccccc} v_1 & \theta_{z1} & v_2 & \theta_{z2} & & \\ d_2 & d_3 & d_5 & d_6 & & \\ \frac{12EI_z}{l^3} & \frac{6EI_z}{l^2} & \frac{-12EI_z}{l^3} & \frac{6EI_z}{l^2} & d_2 & v_1 \\ \frac{6EI_z}{l^2} & \frac{4EI_z}{l} & \frac{-6EI_z}{l^2} & \frac{2EI_z}{l} & d_3 & \theta_{z1} \\ \frac{-12EI_z}{l^3} & -\frac{6EI_z}{l^2} & \frac{12EI_z}{l^3} & \frac{-6EI_z}{l^2} & d_5 & v_2 \\ \frac{6EI_z}{l^2} & \frac{2EI_z}{l} & \frac{-6EI_z}{l^2} & \frac{4EI_z}{l} & d_6 & \theta_{z2} \end{array}$$

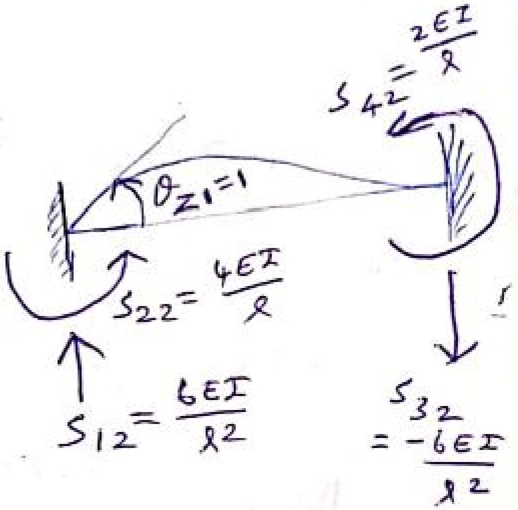
The physical interpretation of the stiffness coefficients given by above equation is illustrated below:

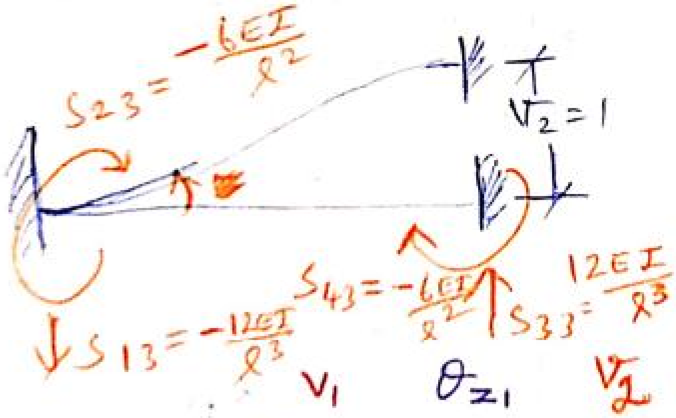


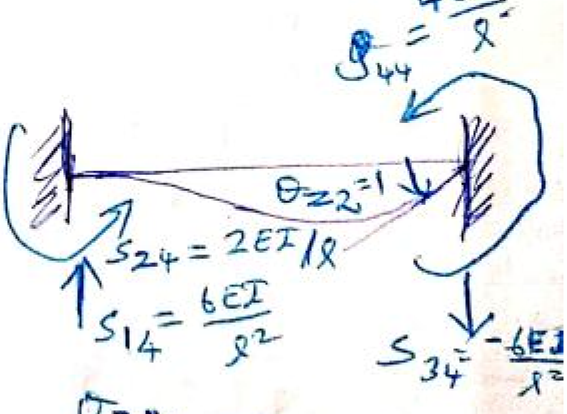


4 Dof : Stiffness matrix (4×4)



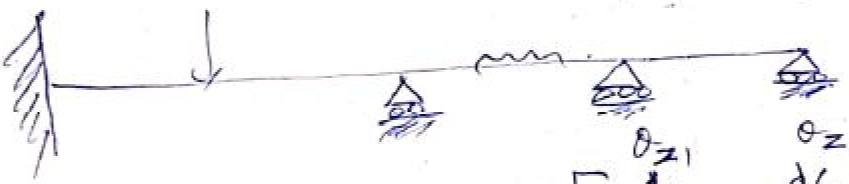


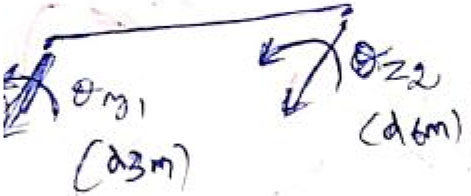




$$
[K_m] = \begin{array}{cccc}
v_1 & \theta_{z_1} & v_2 & \theta_{z_2} \\
\left[\begin{array}{cccc}
\frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\
\frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\
-\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\
\frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l}
\end{array}\right] & & &
\end{array}
$$

Beam: when there is no vertical displacements (when there is no cantilever for displacement at its free end ⟵ I.S) ie hinged/roller/fixed support.





$$
[K_m] = \begin{array}{c|cc|cc}
 & \theta_{z1} & \theta_{z2} & & \\
 & d_3 & d_6 & & \\
\hline
 & \frac{4EI_z}{l} & \frac{2EI_z}{l} & d_3 & \theta_{z1} \\
 & \frac{2EI_z}{l} & \frac{4EI_z}{l} & d_6 & \theta_{z2}
\end{array}
$$

✓

Slope deflection equation

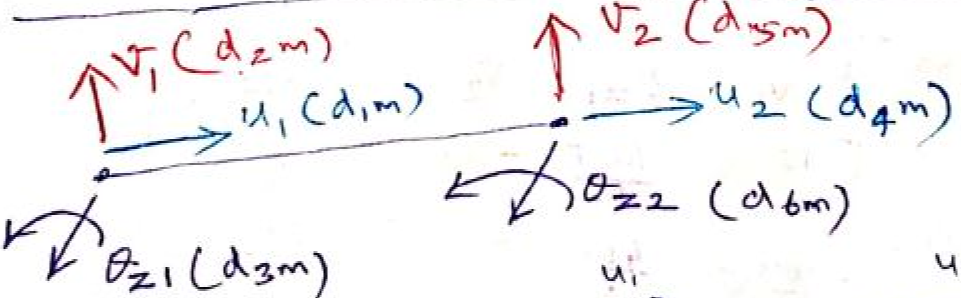
$$M_{AB} = M^F_{AB} + \frac{2EI}{l}(2\theta_A + \theta_B)$$

$$M_{BA} = M^F_{BA} + \frac{2EI}{l}(2\theta_B + \theta_A)$$

In matrix form:

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{Bmatrix} M^F_{AB} \\ M^F_{BA} \end{Bmatrix} + \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix} \begin{Bmatrix} \theta_A \\ \theta_B \end{Bmatrix}$$

✓



For truss element:
(only axial force)

$$
[K_m] = \begin{array}{cc|c}
u_1\,(1) & u_2\,(4) & \\
\hline
\frac{AE}{l} & -\frac{AE}{l} & 1 \\
-\frac{AE}{l} & \frac{AE}{l} & 4
\end{array}
$$

Truss member is 2×2 stiffness matrix.

Stiffness matrix for a 2D beam with 6 DoF

$$[K_m] = \begin{array}{ccccccc} d_{1m} & d_{2m} & d_{3m} & d_{4m} & d_{5m} & d_{6m} & \\ \frac{EA}{l} & 0 & 0 & -\frac{AE}{l} & 0 & 0 & d_{1m} \\ 0 & \frac{12EI_z}{l^3} & \frac{6EI_z}{l^2} & 0 & \frac{-12EI_z}{l^3} & \frac{6EI_z}{l^2} & d_{2m} \\ 0 & \frac{6EI_z}{l^2} & \frac{4EI_z}{l} & 0 & \frac{-6EI_z}{l^2} & \frac{2EI_z}{l} & d_{3m} \\ -\frac{AE}{l} & 0 & 0 & \frac{AE}{l} & 0 & 0 & d_{4m} \\ 0 & \frac{-12EI_z}{l^3} & \frac{-6EI_z}{l^2} & 0 & \frac{12EI_z}{l^3} & -\frac{6EI_z}{l^2} & d_{5m} \\ 0 & \frac{6EI_z}{l^2} & \frac{2EI_z}{l} & 0 & \frac{-6EI_z}{l^2} & \frac{4EI_z}{l} & d_{6m} \end{array}$$

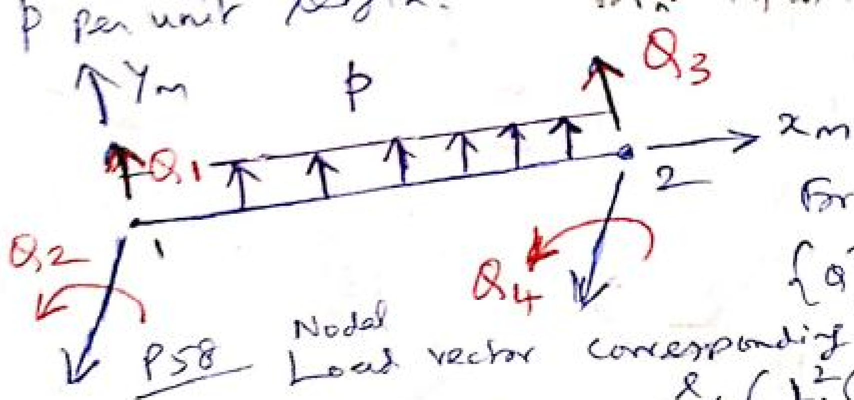
Computation of Element Nodal Load vector

Computation of nodal loads due to lateral loads applied on the beam element is follows. the beam is subjected to an udl (vertical) of p per unit length.

$\iint [N^s]\{p\} ds$ reduced for 2D beam as

P36: Nodal load vector due to Surface Traction.

$$\{Q\} = \int [N^s]^T \{p\} dl$$



For udl of intensity p per unit length,

$$\{Q\} = P \int [N^s]^T dl$$

P58 Nodal Load vector corresponding to 4 DoF, $v_1, \theta_{z1}, v_2, \theta_{z2}$

$$\{Q_m\} = \{Q\} = p \int_0^l \left\{ \begin{array}{l} L_1^2(3-2L_1) \\ L_1^2 L_2 l \\ L_2^2(3-2L_2) \\ -L_1 L_2^2 l \end{array} \right\} dl$$

$$\int L_1^p L_2^q L_3^r \, dl = \frac{p!\,q!\,r!}{(p+q+r+1)!}\, l$$

First Term: $Q_1 = p\int_0^l L_1^2 (3-2L_1)\, dl$

$$= p\int_0^l (3L_1^2 - 2L_1^3)\, dl = p\left[3\cdot\frac{2}{6}l - 2\cdot\frac{6}{24}l\right]_0^l$$

$$= p\left[l - \frac{l}{2}\right] = \boxed{\frac{pl}{2} = Q_1}$$

Second Term: $Q_2 = p\int_0^l L_1^2 L_2^1 l\, dl = p\left[\frac{2!\,1!}{4\times3\times2}\cdot l\right] l$

$$\boxed{Q_2 = \frac{pl^2}{12}}$$

Third Term: $Q_3 = p\int_0^l L_2^2 (3-2L_2)\, dl$

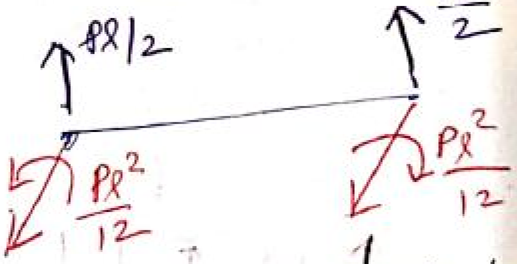
$$= p\int_0^l (3L_2^2 - 2L_2^3)\, dl = p\left[3\cdot\frac{2}{6}l - 2\,\frac{6}{4\times3\times2}l\right]$$

$$\boxed{Q_3 = \frac{pl}{2}}$$

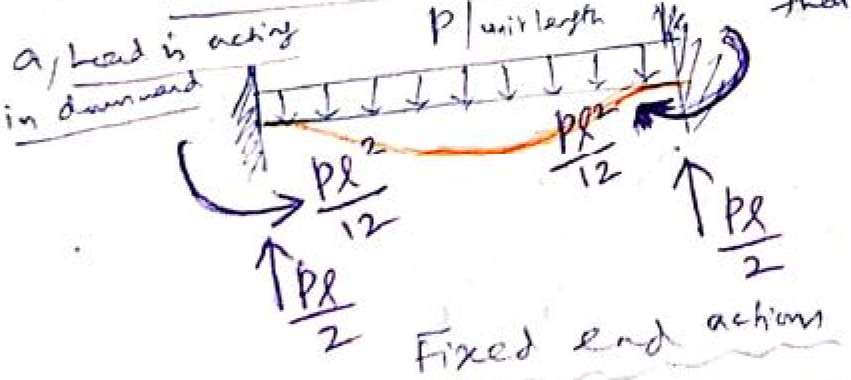
Fourth Term: $Q_4 = p\int_0^l -L_1 L_2^2\, l\, dl$

$$= -p\left[\left(\frac{1!\,2!}{(1+2+1)!} = \frac{2}{4\times3\times2}l\right)\cdot l\right] = \boxed{\frac{-pl^2}{12} = Q_4}$$

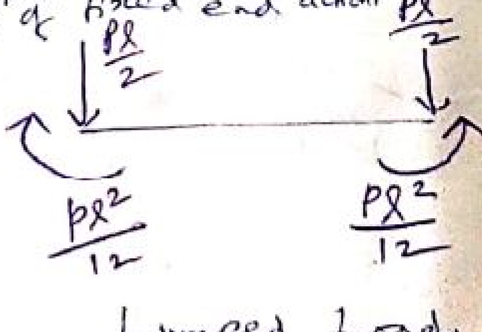
$$\therefore \{Q\} = \left\{\begin{matrix} pl/2 \\ pl^2/12 \\ pl/2 \\ -pl^2/12 \end{matrix}\right\} \begin{matrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{matrix}_{4\times1}$$



Lumped load Vector :- Lumped load is opposite to that of fixed end action

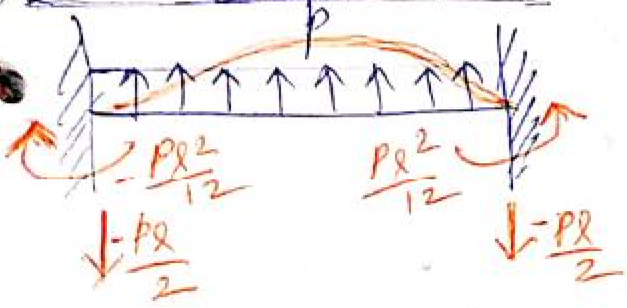


Fixed end actions

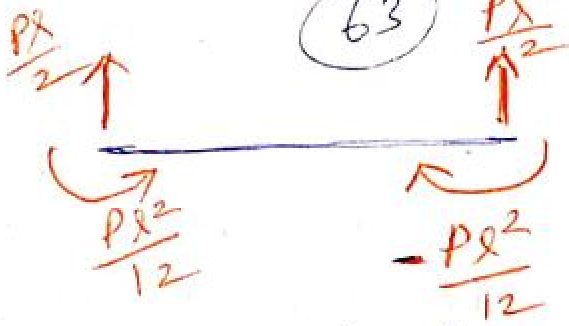


Lumped Loads

(b) Load is acting in upward.



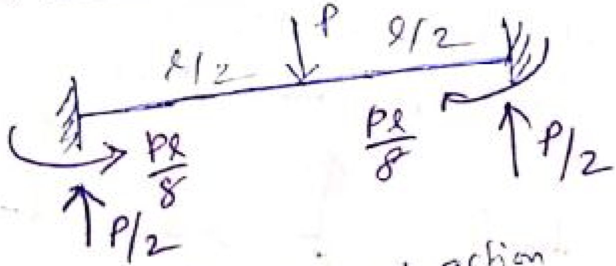
Fixed end action



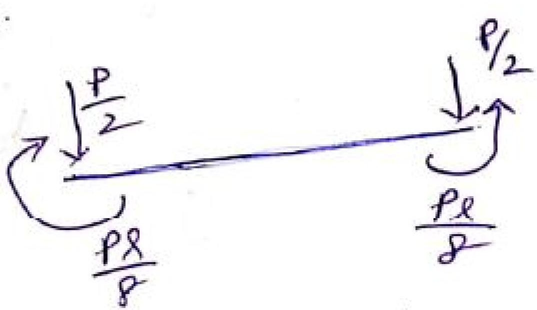
Lumped Loads

It can be seen that the nodal load vector {Q} components are equal to the negative values of fixed end actions of the beam, when the ends are assumed as fixed.

(C) Point load.



Fixed end action



Lumped Loads

Computation of final stress resultants.

- In case of 2 or 3 Dl beam elements, it is necessary to compute the stress resultants at the ends of member for design purposes.

In general, FEA, stress is computed as

$$\{\sigma\} = [C]\,[B]\,\{d\}$$

P12,34

$$\{\sigma\} = [C]\{\varepsilon\}$$
$$\{\varepsilon\} = [B]\{d\}$$

- These stress resultants ~~at~~ corresponds to the dof as axial and shear forces and bending moments
- Hence, as shown in P61 figures, the stiffness coefficients give the value of these actions due to unit displacements.
- The end actions due to end displacements for a member in local axes system can be expressed as $\{S\} = [K_M]\{d_M\}$

Scanned with CamScanner

$$\{S\} = [K_M]\{d_m\}$$

The above stress results we should add the stress resultants due to loads on member under fully restrained conditions. Otherwise for a beam shown in P61, the above equation would give zero values of stress resultants which is incorrect. Thus,

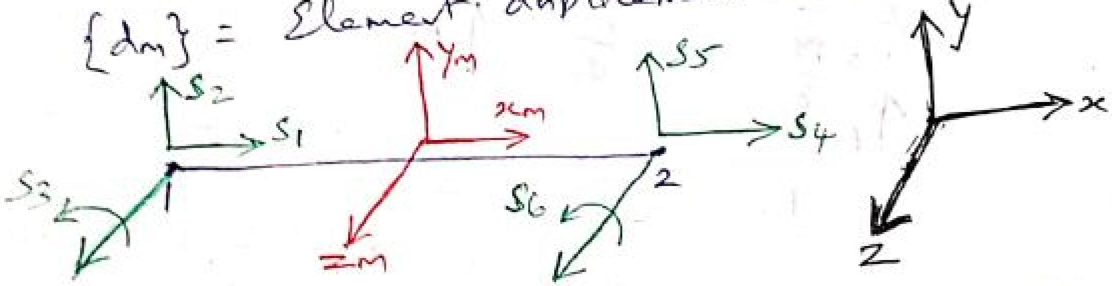
$$\{S\} = \{S_0\} + [K_m]\{d_m\}$$

where $\{S\}$ = Stress resultants at ends of the member.

$\{S_0\}$ = fixed end actions
(Stress resultants corresponding to the nodal dof due to loads on member under fully restrained end conditions).

$[K_m]$ = Element stiffness matrix

$\{d_m\}$ = Element displacement vector.

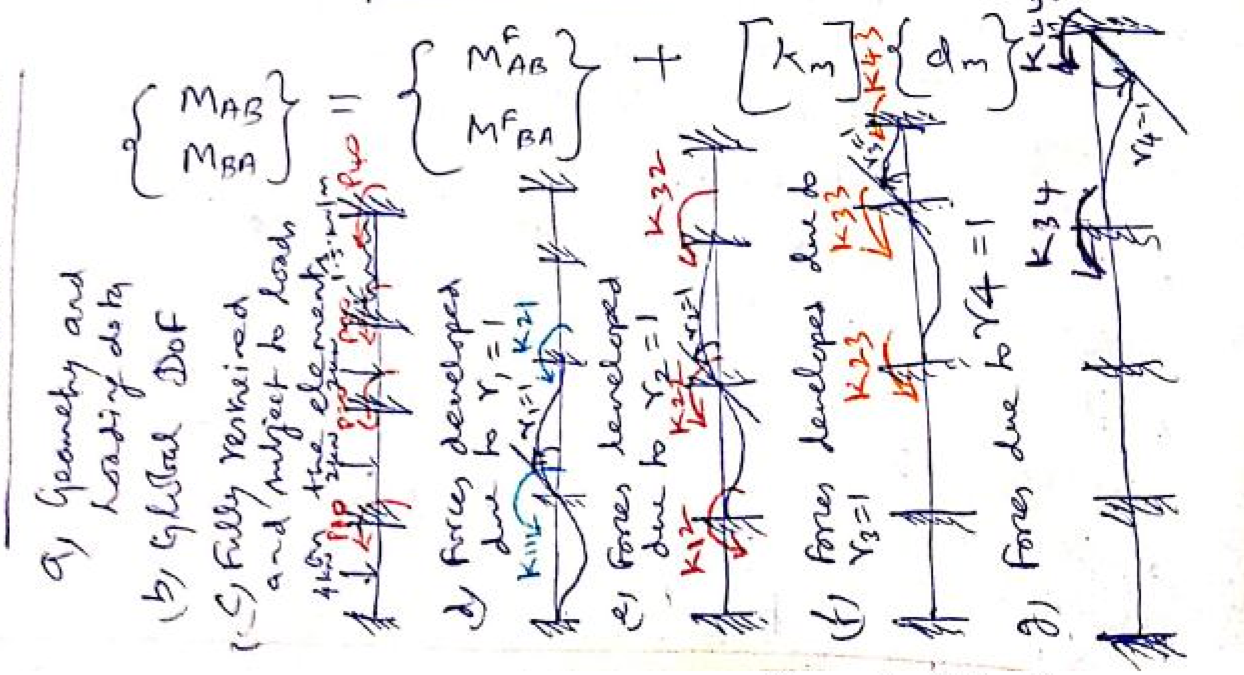


Stress resultants of a 2D beam member

P14/5 CSK

Assembling of Elements – Direct stiffness method.

$$\begin{Bmatrix} M_{AB} \\ M_{BA} \end{Bmatrix} = \begin{Bmatrix} M^f_{AB} \\ M^f_{BA} \end{Bmatrix} + [k_m]\{d_m\}$$

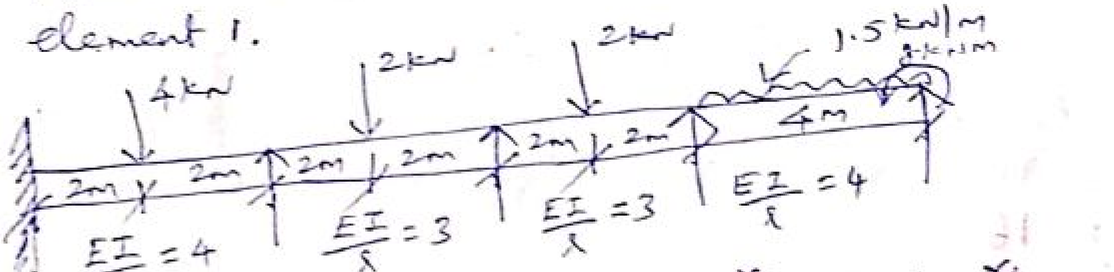


(64)

26/3/20 Thursday

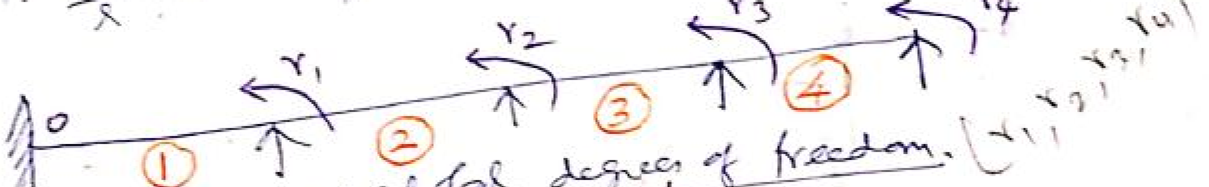
Analyse the continuous beam as shown in figure and evaluate the stress resultant for element 1.

P146 CSK

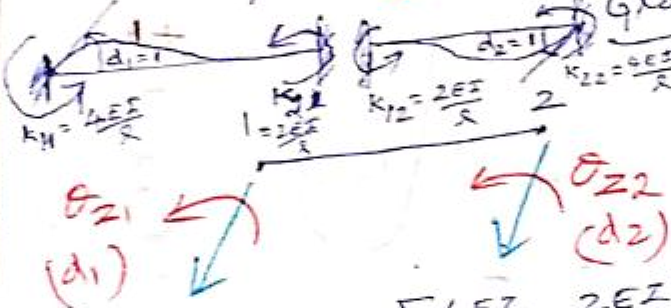


(a)

(b)

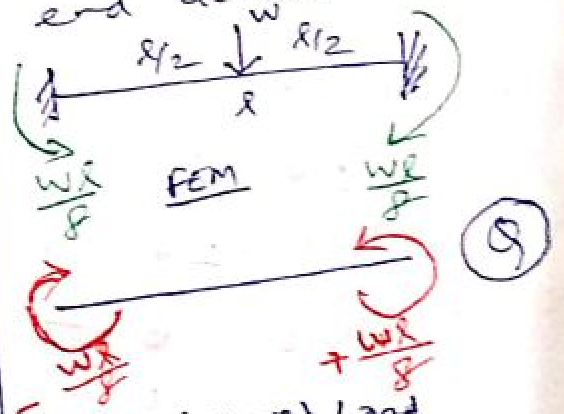


Global degrees of freedom.

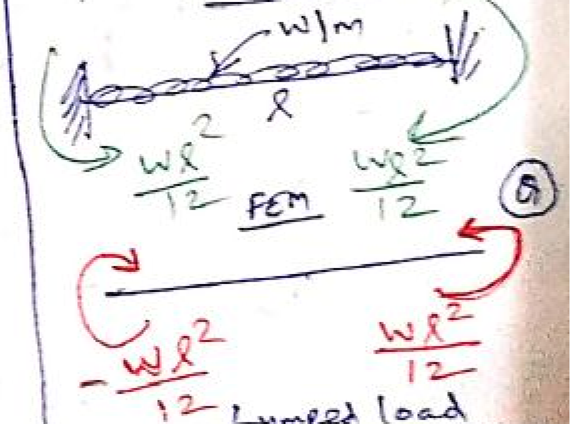


Lumped Load = Consistent load

Lumped load is opposite to fixed end action.



Lumped Load



Lumped load

③
$$[k_m] = \begin{bmatrix} \frac{4EI}{l} & \frac{2EI}{l} \\ \frac{2EI}{l} & \frac{4EI}{l} \end{bmatrix}$$

$[k]\{d\} = \{Q\}$ P36

Element ①

$k_{11} = 4\times4 = 16$ $k_{12} = 2\times4 = 8$

$k_{21} = 8$ $k_{22} = 16$

FEM: $-\frac{4\times4}{8} = -2$ & +2

$$\begin{array}{c|cc|c} & 0 & 1 & \text{g.dof} \\ \hline 1 & \cancel{16} & \cancel{8} & 0 \\ 2 & 8 & 16 & 1 \\ \hline \text{e dof} & 1 & 2 & \end{array} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} \cancel{-2} \\ +2 \end{Bmatrix} \begin{matrix} 0 \\ 1 \end{matrix}$$

FEM: $-\frac{2\times4}{8} = -1$ & 1

Element ②

$$\begin{array}{c|cc|c} & 1 & 2 & \\ \hline 1 & 12 & 6 & 1 \\ 2 & 6 & 12 & 2 \\ \hline \text{e dof} & 1 & 2 & \end{array} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ +1 \end{Bmatrix} \begin{matrix} 1 \\ 2 \end{matrix}$$

Element ③

$$\begin{array}{c|cc|c} & 2 & 3 & \\ \hline 1 & 12 & 6 & 2 \\ 2 & 6 & 12 & 3 \\ \hline \text{e dof} & 1 & 2 & \end{array} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} -1 \\ +1 \end{Bmatrix} \begin{matrix} 2 \\ 3 \end{matrix}$$

Element ④

$\frac{1.5\times16}{12} = 2$ $\frac{4}{3}$

$$\begin{array}{c|cc|c} & 3 & 4 & \\ \hline 1 & 16 & 8 & 3 \\ 2 & 8 & 16 & 4 \\ \hline \text{e dof} & 1 & 2 & \end{array} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} -2 \\ +2 \\ +8 \end{Bmatrix} \begin{matrix} 3 \\ 4 \end{matrix}$$

$\{d\}$ = vector of nodal displt.

The element DoF d_1 & d_2 correspond to 0 and r_1.

③ $$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix}$$

** $k_{11}, k_{21}, k_{12}, k_{22}$ refer to forces developed @ DoF.

Note: Moment applied along degree of freedom 2 can be straightaway added.

$$[K]\{r\} = \{R\}$$

where
$[K]$ is the stiffness matrix or global stiffness matrix
$\{r\}$ is displacement vector consisting of global DoF.
$\{P\}$ is load vector corresponding to DoF (r).

Assembling :

wrt 63b superimposing (e) to (g)

$$\begin{array}{c|cccc} & r_1 & r_2 & r_3 & r_4 \\ \hline & \substack{16\\12} & 6 & & \\ & 6 & \substack{12\\12} & 6 & \\ & & 6 & \substack{12\\16} & 8 \\ & & & 8 & 16 \end{array} \Leftarrow \text{g.dof} \quad \begin{array}{c} \text{g.dof} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} +2 \\ -1 \\ +1 \\ -1 \\ +1 \\ -2 \\ +2 \\ +8 \end{Bmatrix} \begin{array}{c} \text{gdof} \\ \downarrow \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Thus, the overall equation of equilibrium for the structure is

$$\begin{bmatrix} 28 & 6 & 0 & 0 \\ 6 & 24 & 6 & 0 \\ 0 & 6 & 28 & 8 \\ 0 & 0 & 8 & 16 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 10 \end{Bmatrix}$$

Assemblage of element by direct stiffness method is physically interpreted. This process can be easily programmed.

Solving by Gaussian Elimination.

$$r_4 = 0.7576$$

$$r_3 = \frac{-0.9434 - 8(0.7576)}{26.415} = -0.2652$$

$$r_2 = \frac{-0.2143 - (6\times(-0.2652)) - 0}{22.714} = 0.0606$$

$$r_1 = \frac{1 - (6\times 0.0606) - 0 - 0}{28} = 0.0227$$

P160 → Decomposition method ✓

$$\begin{bmatrix} 28 & 6 & 0 & 0 \\ 6 & 24 & 6 & 0 \\ 0 & 6 & 28 & 8 \\ 0 & 0 & 8 & 16 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} +1 \\ 0 \\ -1 \\ 10 \end{Bmatrix}$$

(i) Elimination of r_1 from all but the first equation by subtracting l_{i1} times the first equation from ith eqn.

$$l_{i1} = \frac{k_{i1}}{k_{11}} = \frac{k_{i1}}{28} \quad i = 2,3,4 \quad l_{21} = \frac{k_{21}}{k_{11}} = \frac{6}{28} = 0.2143$$

$k_{31} = 0 - 0 \times 28 = 0$

$k_{32} = 6 - 0 \times 6 = 6$

$k_{33} = 28 - 0 \times 0 = 28$

$k_{34} = 8 - 0 \times 0$

$l_{31} = \frac{k_{31}}{k_{11}} = 0$

$l_{41} = \frac{k_{41}}{k_{11}} = 0$

$k_{41} = 0 - 0 \times 28 = 0$

$k_{42} = 0 - 0 \times 6 = 0$

$k_{43} = 8 - 0 \times 0 = 8$

$k_{44} = 16 - 0 \times 0 = 16$

$Q_4 = 10 - 0 \times 1 = 10$

$$\begin{bmatrix} 28 & 6 & 0 & 0 \\ 0 & 22.714 & 6^{(1)} & 0 \\ 0 & 6 & 28 & 8 \\ 0 & 0 & 8 & 16 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ -0.2143 \\ -1.0 \\ 10.0 \end{Bmatrix}$$

$k_{21} = 6 - 0.2143 \times 28 = 0$

$k_{22} = 24 - 0.2143 \times 6 = 22.71$

$k_{23} = 6 - 0.2143 \times 0 = 6$

$k_{24} = 0 - '' \times 0 = 0$

$Q_2 = 0 - 0.2143 \times 1$

(ii) Subtracting l_{i2} times the second equation from third and fourth equation, r_2 is eliminated.

$$l_{i2} = l_{32} = \frac{k^{i}_{32}{}^{(2)}}{k_{22}} = \frac{6}{22.714} = 0.26415$$

$i = 3,4$

$$l_{42} = \frac{k_{42}}{k_{22}} = 0$$

$k_{32} = 6 - 0.26415 \times 22.714 = 0$

$k_{33} = 28 - '' \times 6 = 26.41$

$k_{34} = 8 - '' \times 0 = 8$

$Q_3 = -1 + '' \times 0.2143$
$= -0.9434$

$k_{42} = 0 - 0 \times 22.714$
$= 0$

$k_{43} = 8 - 0 \times 6 = 8$

$k_{44} = 16 - 0 \times 0$
$= 16$

$Q_4 = 10 - 0 \times (-0.2143) = 10$

$$\begin{bmatrix} 28 & 6 & 0 & 0 \\ 0 & 22.714 & 6 & 0 \\ 0 & 0 & 26.415 & 8 \\ 0 & 0 & 8^{(3)} & 16 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ -0.2143 \\ -0.9434 \\ 10 \end{Bmatrix}$$

(iii) Subtracting l_{i3} times the third equation from fourth equation r_3 is eliminated.

$$l_{i3} = \frac{k_{i3}}{k_{33}} = l_{43} = \frac{k_{43}^{(3)}}{k_{33}} = \frac{8}{26.415} = 0.302858$$

$i = 4$

$k_{43} = 8 - l \times 26.415$
$= 0$

$$\begin{bmatrix} 28 & 6 & 0 & 0 \\ 0 & 22.714 & 6 & 0 \\ 0 & 0 & 26.415 & 8 \\ 0 & 0 & 0 & 13.577 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 1.0 \\ -0.2143^{(1)} \\ -0.9434^{(3)} \\ 10.2857 \end{Bmatrix}$$

$k_{44} = 16 - 0.302858 \times 8$
$= 13.577$

$Q_4 = 10 - 0.302858 \times (-0.9434)$
$=$

$$r_4 = 10.2857 / 13.577 = 0.7576$$

$\longleftarrow r_3, r_2, r_1$

Member end actions.

$$\{S\} = \{S_o\} + [k_m]\{d_m\}$$

{S} = Member end action.

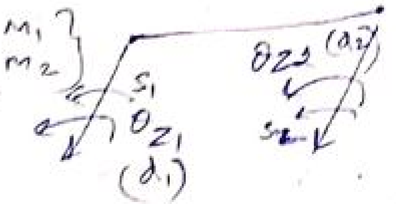
{S_o} = Fixed end action.

[k_m] : Element Stiffness matrix.

{d_m} : Element dat

Element ①

$$\{S\} = \begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix} = \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}$$



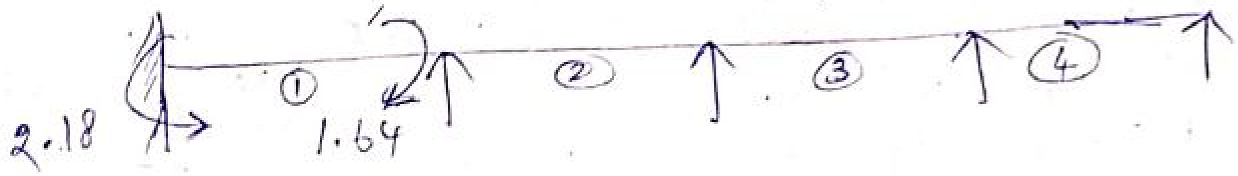
$$\{S_o\} = \begin{Bmatrix} +2 \\ -2 \end{Bmatrix}$$

$$\{k_1\} = \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix}$$

$$\{d_{1m}\} = \begin{Bmatrix} 0 \\ r_1 \end{Bmatrix}$$

$$\begin{Bmatrix} S_1 \\ S_2 \end{Bmatrix} = \begin{Bmatrix} +2 \\ -2 \end{Bmatrix} + \begin{bmatrix} 16 & 8 \\ 8 & 16 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.0227 \end{Bmatrix}$$

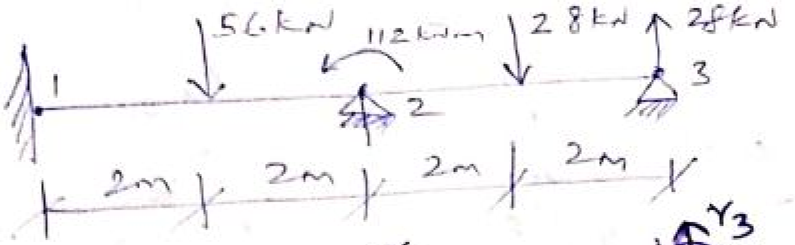
$$= \begin{Bmatrix} +2.1816 \\ -1.6368 \end{Bmatrix}$$

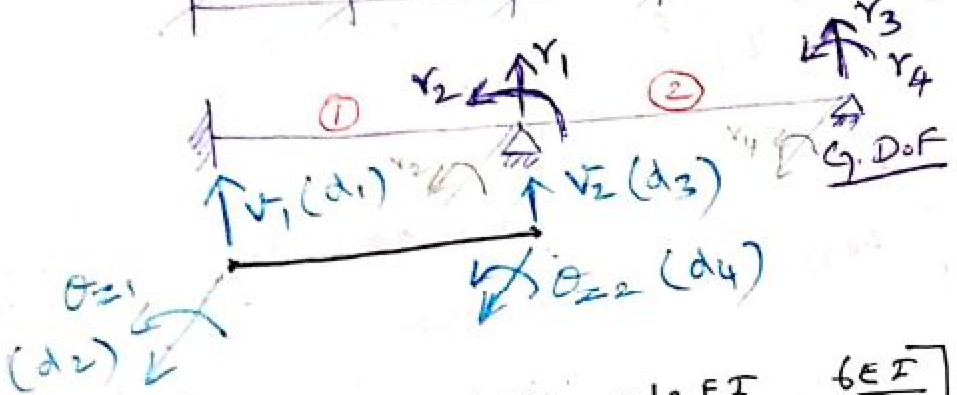


27/3

⑥⑥

Analyse the continuous beam and compute the support reaction at node 3 by adopting a dof along the reaction also evaluate the reaction by using boundary element concept. Draw SFD and BMD. Take EI = 400 units.





P61

$$k_m = \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ \frac{-12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & \frac{-6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

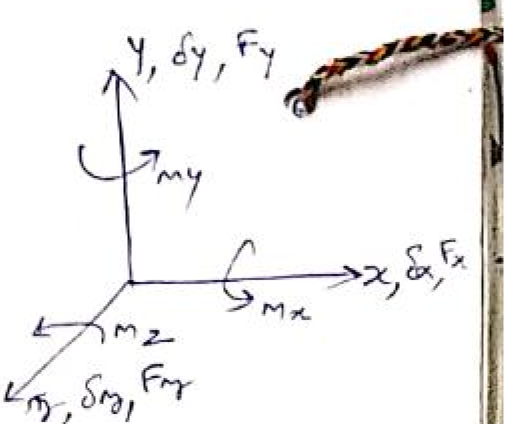
Elements ① and ② :

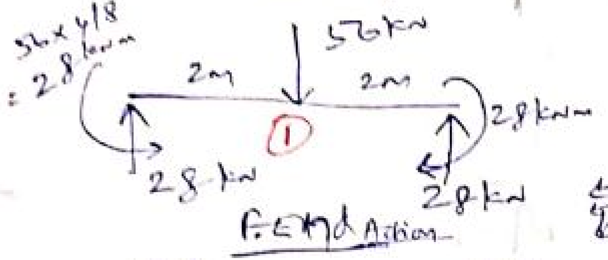
$$\begin{array}{cccccc} \overset{1}{0} & \overset{2}{0} & \overset{3}{1} & \overset{4}{2} & \leftarrow ① & \leftarrow ② \text{ G.D.oF} \end{array}$$

$$k_m = \left[\begin{array}{cccc|cc} 75 & 150 & -75 & 150 & 0 & 1 \\ 150 & 400 & -150 & 200 & 0 & 2 \\ -75 & -150 & \boxed{75} & \boxed{-150} & 1 & 3 \\ 150 & 200 & \boxed{-150} & \boxed{400} & 2 & 4 \end{array}\right]$$

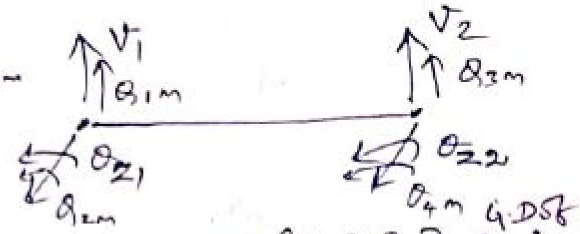
Scanned with CamScanner

$$[k] = \begin{array}{c|cccc|c} & 1 & 2 & 3 & 4 & \leftarrow \text{GDof} \\ \hline & \substack{75 \\ 75} & \substack{-150 \\ 150} & -75 & 150 & 1 \\ & \substack{-150 \\ 150} & \substack{400 \\ 400} & -150 & 200 & 2 \\ & -75 & -150 & 75 & -150 & 3 \\ & 150 & 200 & -150 & 400 & 4 \end{array}$$



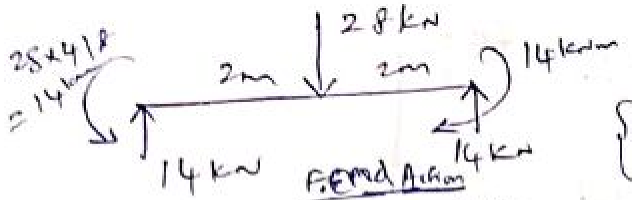


F.E.M.d Action



Lumped load

$$\{Q_1\} = \begin{array}{c} \\ \end{array}\begin{Bmatrix} -28 \\ -28 \\ \boxed{-28} \\ \boxed{28} \end{Bmatrix} \begin{array}{c} 0 \\ 0 \\ 1 \\ 2 \end{array} \quad \text{G.DoF} \downarrow$$



F.E.M.d Action

Lumped load

$$\{Q_2\} = \begin{Bmatrix} -14 \\ -14 \\ -14 \\ 14 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \text{GDof} \downarrow$$

$$Q = \begin{Bmatrix} -28-14 = -42 \\ 28-14+112 = 126 \\ -14+28 = 14 \\ 14 = 14 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \quad \text{GDof}$$

Equilibrium Equation: $[k]\{r\} = \{Q\}$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \end{array}$$
$$\begin{bmatrix} 150 & 0 & -75 & 150 \\ 0 & 800 & -150 & 200 \\ -75 & -150 & 75 & -150 \\ 150 & 200 & -150 & 400 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} -42 \\ 126 \\ 14 \\ 14 \end{Bmatrix}$$

Imposing boundary conditions, $\boxed{r_1 = r_3 = 0}$

$$\begin{Bmatrix} 800 & 200 \\ 200 & 400 \end{Bmatrix} \begin{Bmatrix} r_2 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} 126 \\ 14 \end{Bmatrix}$$

Solving. $r_2 = 0.17$ $r_4 = -0.05$

$800 r_2 + 200 r_4 = 126$, $200 r_2 + 400 r_4 = 14$

$\times 4$ $\quad 800 r_2 + 1600 r_4 = 56$

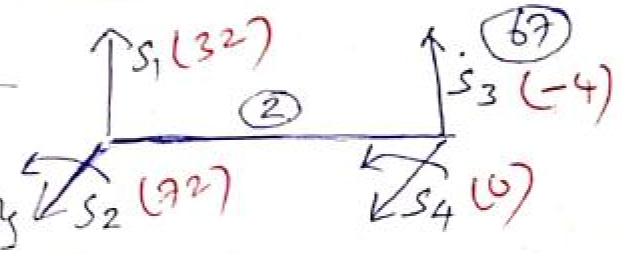
$-1400 r_4 = 70$

$r_4 = -0.05$

$\therefore$ or $r_2 = 0.17$

Member end actions

Member ②

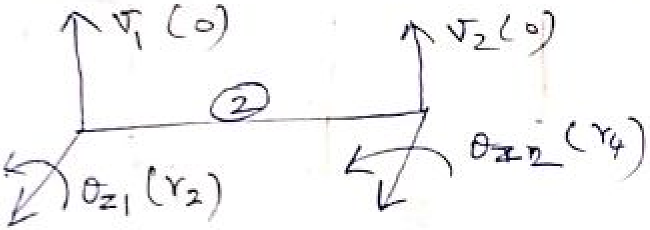


$$\{S\} = \{S_0\} + [k_m]\{d_m\}$$

$\{S_0\}$ → Fixed end actions.

$[k_m]$ → Element Stiffness Matrix

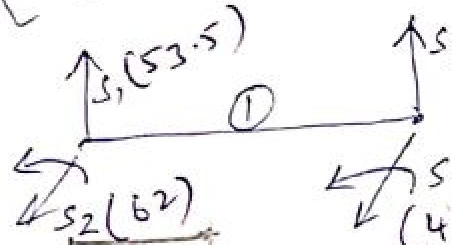
$\{d_m\}$ → Element DoF.

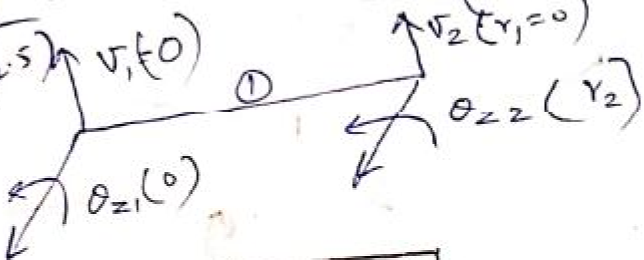


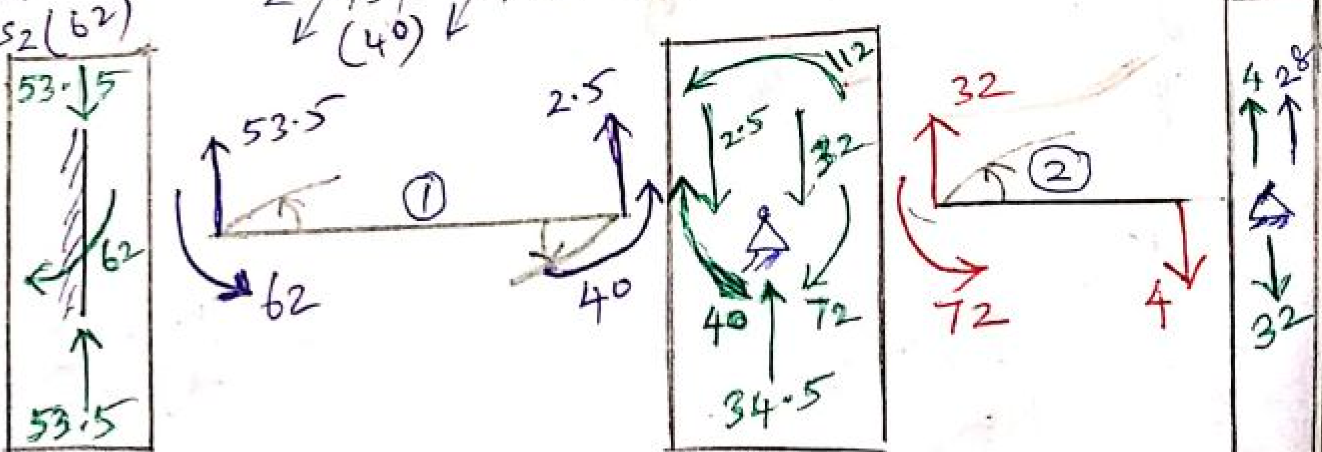
$$\begin{Bmatrix} S_1\\ S_2\\ S_3\\ S_4\end{Bmatrix} = \{S\} = \begin{Bmatrix}14\\14\\14\\-14\end{Bmatrix} + \begin{bmatrix}75&150&-75&150\\150&400&-150&200\\-75&-150&75&-150\\150&200&-150&400\end{bmatrix}\begin{Bmatrix}0\\0.17\\0\\-0.05\end{Bmatrix} = \begin{Bmatrix}32\\72\\-4\\0\end{Bmatrix}\begin{matrix}S_1\\S_2\\S_3\\S_4\end{matrix}$$

Member ①

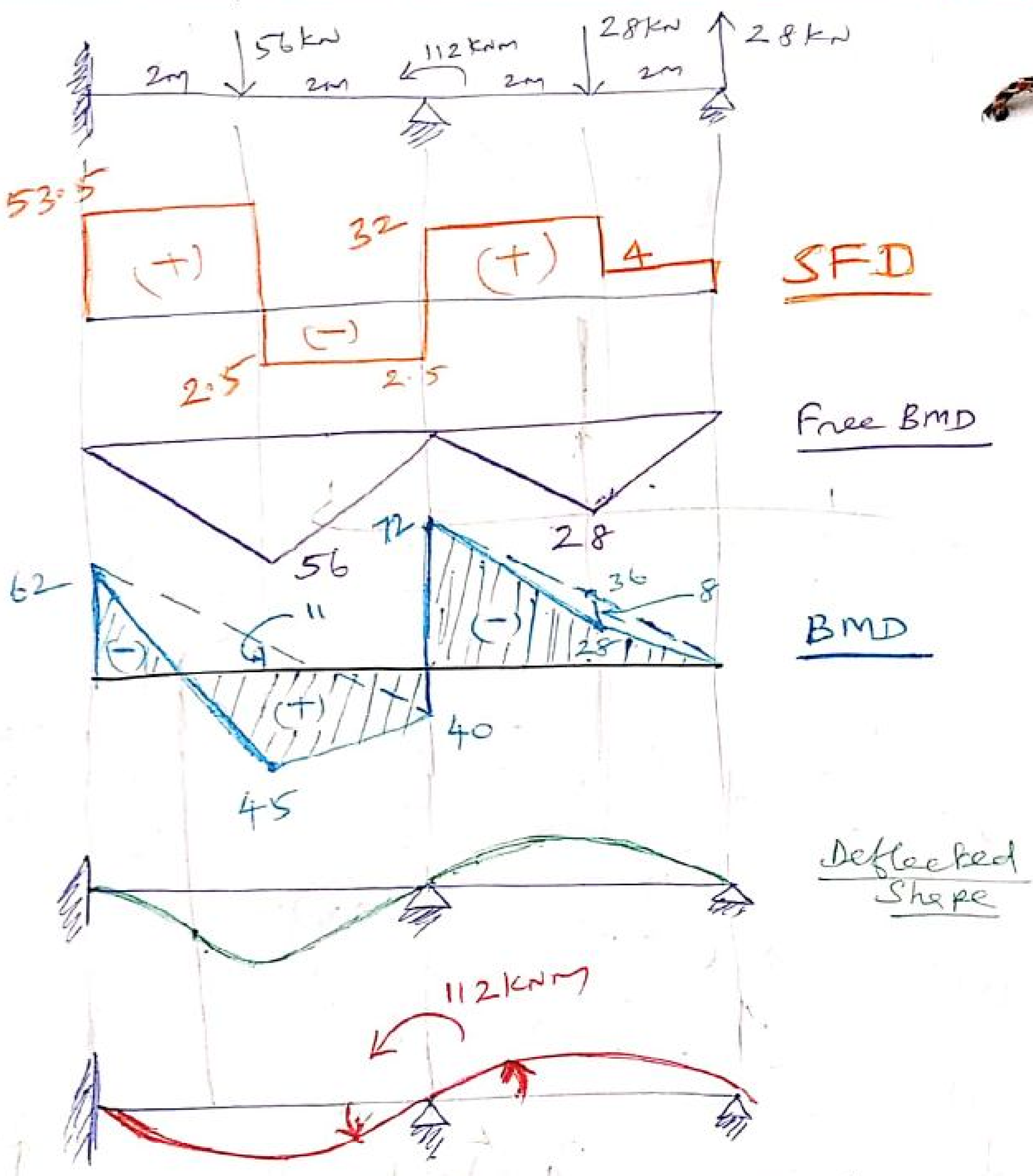
$$\begin{Bmatrix} S_1\\ S_2\\ S_3\\ S_4\end{Bmatrix} = \{S\} = \begin{Bmatrix}28\\28\\28\\-28\end{Bmatrix} + \begin{bmatrix}75&150&-75&150\\150&400&-150&200\\-75&-150&75&-150\\150&200&-150&400\end{bmatrix}\begin{Bmatrix}0\\0\\0\\0.17\end{Bmatrix} = \begin{Bmatrix}53.5\\62\\2.5\\40\end{Bmatrix}\begin{matrix}S_1\\S_2\\S_3\\S_4\end{matrix}$$

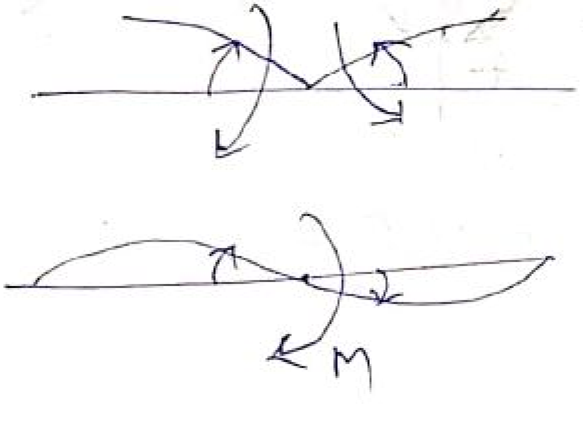


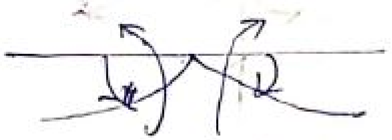




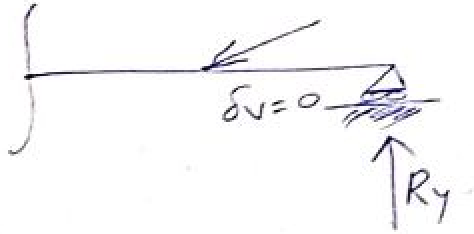
The reaction at node ③ = 4 + 28 = 32 kN ↓

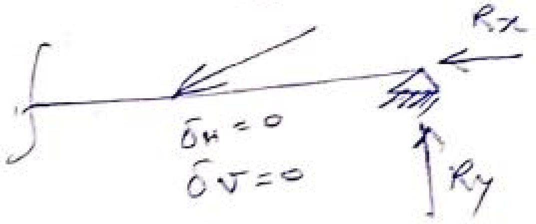

56kN
112kNm
28kN
28kN
2m
2m
2m
2m
53.5
(+)
(−)
2.5
2.5
32
(+)
4
SFD
Free BMD
56
28
72
62
(−)
11
(+)
(−)
36
8
28
40
45
BMD
Deflected Shape
112kNm

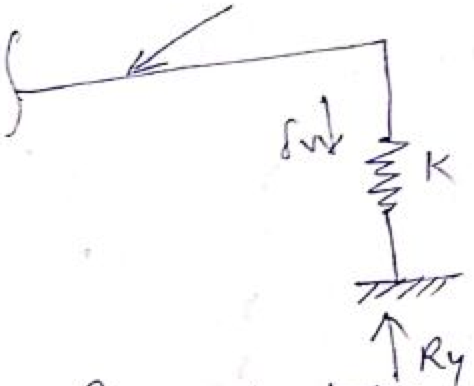

M

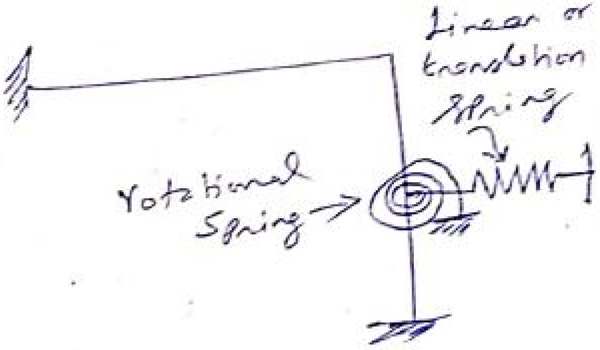


Boundry Element [Spring Element]

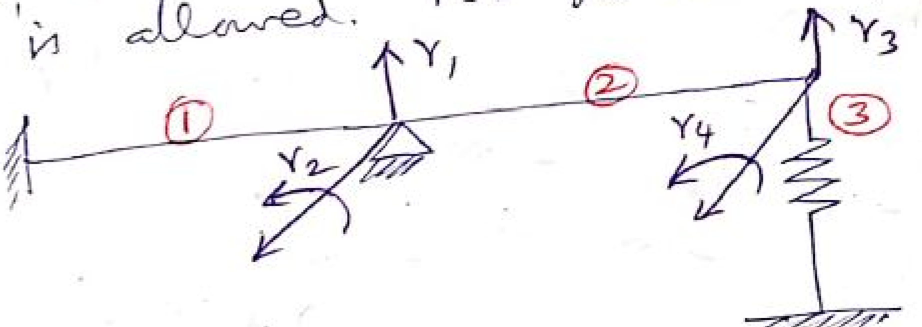








Note: In water tank design, limit state design is not permitted. Only working stress method is permitted. Because, in the limit state design 0.3mm crack width is allowed. PSF for conc = 1.5, steel: 1.15.



The main purpose of the boundry element is to apply displacement boundry condition and compute the values of support reaction.

The boundry element is the spring element with axial and torsional stiffness to get the value of the reaction in the direction of the particular degree of freedom, very high stiffness coefficient is added in the corresponding diagonal coefficient.

This yields a very small but finite displacement along the degree of freedom which when multiplied by imposed stiffness gives the desired reactions.

Introduce a spring (boundary element) of very high stiffness value of 10^5 kN/m.

$$[k_s] = [k] = \overset{\overset{\text{g.dof}}{\downarrow}}{\underset{3}{}}\left[10^5\right]_3$$

(P166)

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & \\ \end{array} \\ \left[\begin{array}{cccc|c} \cancel{150} & \cancel{0} & \cancel{-75} & \cancel{150} & 1 \\ \cancel{0} & 800 & -150 & 200 & 2 \\ \cancel{-75} & -100 & (75+10^5) & -150 & 3 \\ \cancel{150} & 200 & -150 & 400 & 4 \end{array}\right] \end{array} \begin{Bmatrix} \cancel{r_1} \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{Bmatrix} \cancel{-42} \\ 126 \\ 14 \\ 14 \end{Bmatrix}$$

Imposing the boundary condition, $r_1 = 0$

$$\begin{Bmatrix} r_2 \\ r_3 \\ r_4 \end{Bmatrix} = \begin{bmatrix} 800 & -150 & 200 \\ -150 & 75+10^5 & -150 \\ 200 & -150 & 400 \end{bmatrix}^{-1} \begin{Bmatrix} 126 \\ 14 \\ 14 \end{Bmatrix} \quad \Bigg| \quad A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = 800\left(100075 \times 400 - (-150)^2\right) + 150\left((-150 \times 400) + (200 \times 150)\right) + 200\left((-150)^2 - (100075 \times 200)\right) = 3.2006 \times 10^{10} - 4.5 \times 10^6 - 3.9985 \times 10^9 = 2.8003 \times 10^{10}.$$

$$C = \begin{bmatrix} +\begin{vmatrix} 100075 & -150 \\ -150 & 400 \end{vmatrix} & -\begin{vmatrix} -150 & -150 \\ 200 & 400 \end{vmatrix} & +\begin{vmatrix} -150 & 100075 \\ 200 & -150 \end{vmatrix} \\ -\begin{vmatrix} -150 & 200 \\ -150 & 400 \end{vmatrix} & +\begin{vmatrix} 800 & 200 \\ 200 & 400 \end{vmatrix} & -\begin{vmatrix} 800 & -150 \\ 200 & -150 \end{vmatrix} \\ +\begin{vmatrix} -150 & 200 \\ 100075 & -150 \end{vmatrix} & -\begin{vmatrix} 800 & 200 \\ -150 & -150 \end{vmatrix} & +\begin{vmatrix} 800 & -150 \\ -150 & 100075 \end{vmatrix} \end{bmatrix}$$

$$[c] = [c]^T$$

$$[A]^{-1} = \frac{[c]^T}{|A|} = \begin{bmatrix} 0.0014286862 & 0.0000010713 & -0.0007139414 \\ 0.0000010713 & 0.0000099989 & 0.0000032139 \\ -0.0007139414 & 0.0000032139 & 0.0028581759 \end{bmatrix}$$

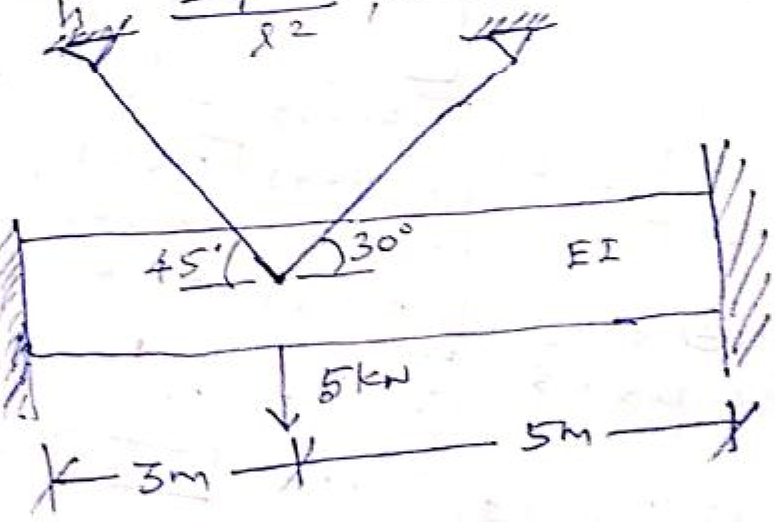
$$\begin{Bmatrix} r_2 \\ r_3 \\ r_4 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 126 \\ 14 \\ 14 \end{Bmatrix} \Rightarrow \begin{array}{l} r_2 = 0.170034298 \\ r_3 = 0.000319963 \\ r_4 = -0.0498971592 \end{array}$$

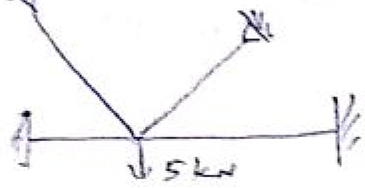
Reaction at Node 3 $= r_3 \times$ imposed stiffness
$= 0.0003199 \times 10^5$
$= 31.99 \simeq 32$ kN ↓

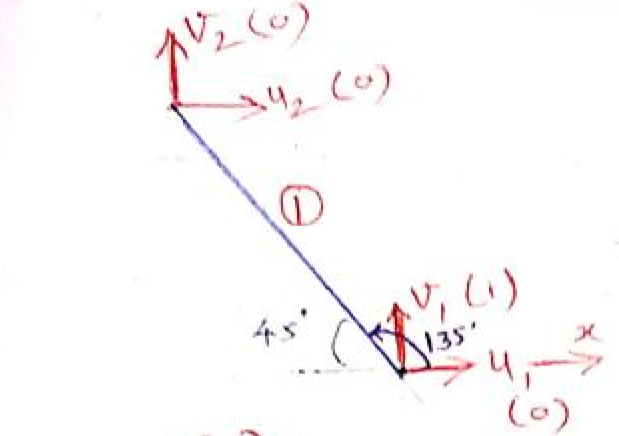
<u>Note:</u> Reaction is the opposite direction of displacement.

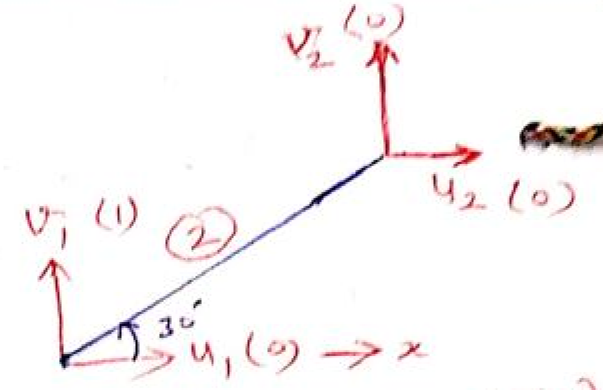
28/2/20
(2)

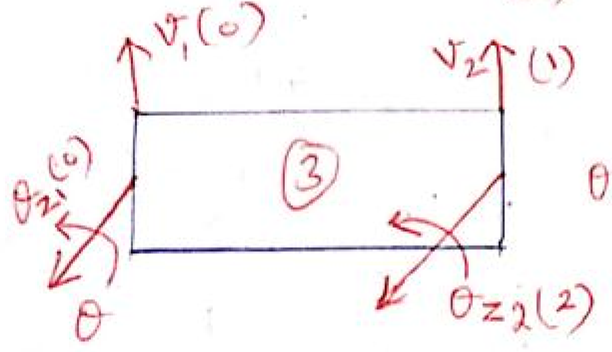
A beam is connected to two numbers of truss bars as shown in figure. The axial deformation of the beam can be neglected. The axial rigidity of the truss is $\frac{24EI}{l^2}$, where $l = 3$m.

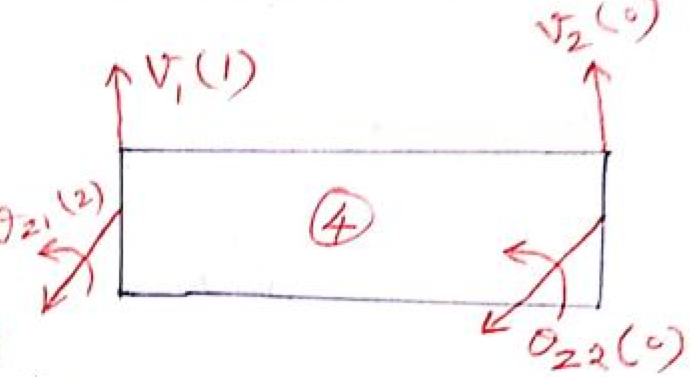












Element ① : $\boxed{l = 3m}$

$AE = \cancel{24EI}\ \frac{24EI}{l^2}$

$\frac{AE}{l} = \frac{24EI}{l^3}$; $\theta = 135°$, $C_x = \cos\theta = -0.707$, $C_y = \sin\theta = 0.707$

$= 0.89EI$

P52b

$$[k_m] = \frac{AE}{l}\left[\begin{array}{cc|cc} C_x^2 & C_xC_y & -C_x^2 & -C_xC_y \\ C_xC_y & C_y^2 & -C_xC_y & -C_y^2 \\ \hline -C_x^2 & -C_xC_y & C_x^2 & C_xC_y \\ -C_xC_y & -C_y^2 & C_xC_y & C_y^2 \end{array}\right]$$

$$[k_1] = EI \begin{array}{cccc|c} 0 & 1 & 0 & 0 & \leftarrow GDOF \\ 0.445 & -0.445 & -0.445 & 0.445 & 0 \\ -0.445 & \boxed{0.445} & 0.445 & -0.445 & 1 \\ -0.445 & 0.445 & 0.445 & -0.445 & 0 \\ 0.445 & -0.445 & -0.445 & 0.445 & 0 \end{array}$$

Element ② $\theta = 30°$

$C_x = 0.866$ $C_y = 0.5$

$$[k_2] = EI \begin{array}{cccc|c} 0 & 1 & 0 & 0 & \leftarrow gdof \\ 0.6675 & 0.385 & -0.6675 & -0.385 & 0 \\ 0.385 & \boxed{0.223} & -0.385 & -0.223 & 1 \\ -0.6675 & -0.385 & 0.6675 & 0.385 & 0 \\ -0.385 & -0.223 & 0.385 & 0.223 & 0 \end{array}$$

Element ③ $\boxed{l = 3m}$

(P61)

$$[k_m] = \begin{bmatrix} \frac{12EI}{l^3} & \frac{6EI}{l^2} & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{4EI}{l} & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ \frac{6EI}{l^2} & \frac{2EI}{l} & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix} \leftarrow \text{g dof} \downarrow$$

$$[k_3] = EI \begin{array}{c} \begin{array}{cccc} 0 & 0 & 1 & 2 \end{array} \\ \left[\begin{array}{cccc} 0.444 & 0.667 & -0.444 & 0.667 \\ 0.667 & 1.333 & -0.667 & 0.667 \\ -0.444 & -0.667 & 0.444 & -0.667 \\ 0.667 & 0.667 & -0.667 & 1.333 \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 0 \\ 1 \\ 2 \end{array}$$

Element ④ : $\boxed{l = 5m}$ $\leftarrow$ g dof $\downarrow$

$$[k_4] = EI \begin{array}{c} \begin{array}{cccc} 1 & 2 & 0 & 0 \end{array} \\ \left[\begin{array}{cccc} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 0 \\ 0 \end{array}$$

Assembling, $[k]\{r\} = \{Q\}$

$$EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \left[\begin{array}{c|c} \begin{array}{c} 0.445 \\ 0.223 \\ 0.444 \\ 0.096 \end{array} & \begin{array}{c} -0.667 \\ 0.24 \end{array} \\ \hline \begin{array}{c} -0.667 \\ 0.24 \end{array} & \begin{array}{c} 1.333 \\ 0.80 \end{array} \end{array}\right] \end{array} \begin{array}{c} 1 \\ \\ 2 \end{array} \cdot \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} -5.0 \\ 0 \end{Bmatrix}$$

$-0.427 r_1 + 2.133 r_2 = 0$

$1.20\cancel{8}r_1 - 0.427 r_2 = -\frac{5.0}{EI},$

$-1.20\cancel{8}r_1 + 6.034 r_2 = 0$

$5.607 r_2 = -5$

$r_2 = \frac{-5}{5.607} = \frac{-0.892}{EI}$

$r_1 = -4.456/EI$

$$EI \begin{bmatrix} 1.208 & -0.427 \\ -0.427 & 2.133 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \end{Bmatrix} = \begin{Bmatrix} -5.0 \\ 0 \end{Bmatrix}$$

$$r_1 = \frac{-4.456}{EI}, \quad r_2 = \frac{-0.892}{EI}$$

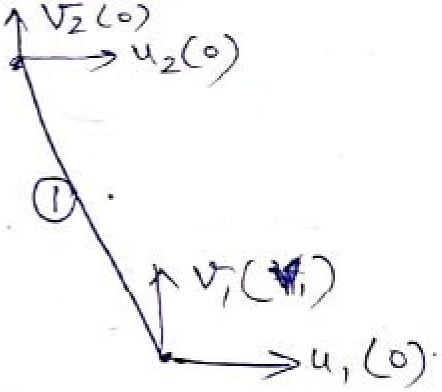
$$\{S\} = [k_m]\{d_m\} + \{S_0\}$$

$\{S_0\}$: Fixed end actions (due to member loads)

(Beam) → {0}.

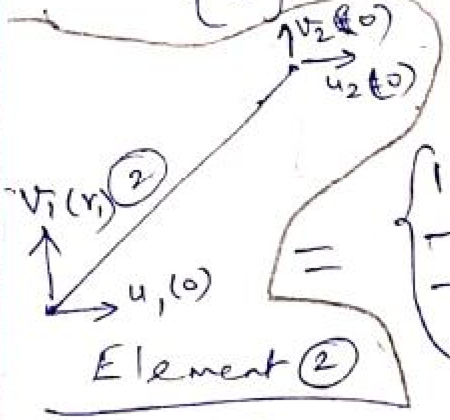
frame

Element ①



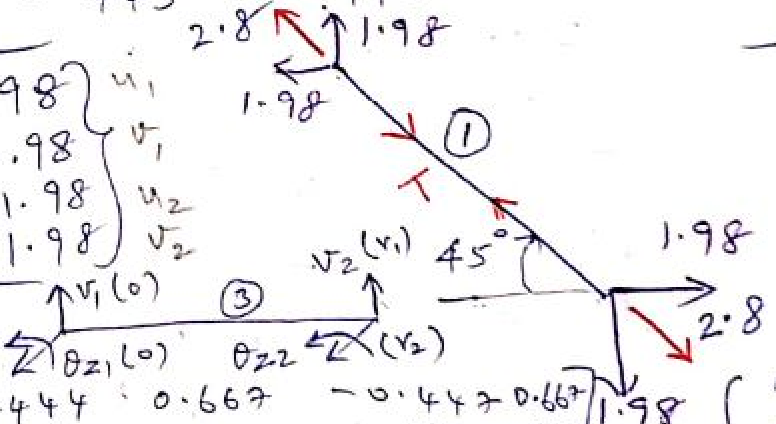
$$\{d_{1m}\} = \begin{Bmatrix} 0 \\ r_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{-4.456}{EI} \\ 0 \\ 0 \end{Bmatrix}$$

$$\{S\} = EI \begin{bmatrix} -0.445 & -0.445 & -0.445 & 0.445 \\ -0.445 & 0.445 & 0.445 & -0.445 \\ -0.445 & 0.445 & 0.445 & -0.445 \\ 0.445 & -0.445 & -0.445 & 0.445 \end{bmatrix} \begin{Bmatrix} 0 \\ -4.456 \\ 0 \\ 0 \end{Bmatrix} r_1$$



Element ②

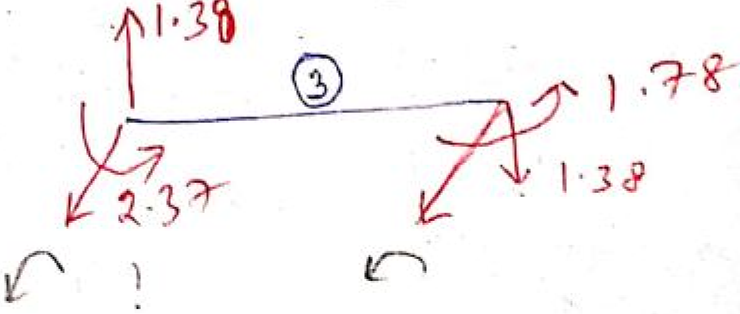
$$= \begin{Bmatrix} 1.98 \\ -1.98 \\ -1.98 \\ 1.98 \end{Bmatrix} \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{matrix}$$



Element ③

$$\{S\} = EI \begin{bmatrix} 0.444 & 0.667 & -0.447 & 0.667 \\ 0.667 & 1.333 & -0.667 & 0.667 \\ -0.444 & -0.667 & 0.444 & -0.667 \\ 0.667 & 0.667 & -0.667 & 1.333 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{-4.456}{EI} \\ \frac{-0.892}{EI} \end{Bmatrix} \begin{matrix} \\ \\ r_1 \\ r_2 \end{matrix}$$

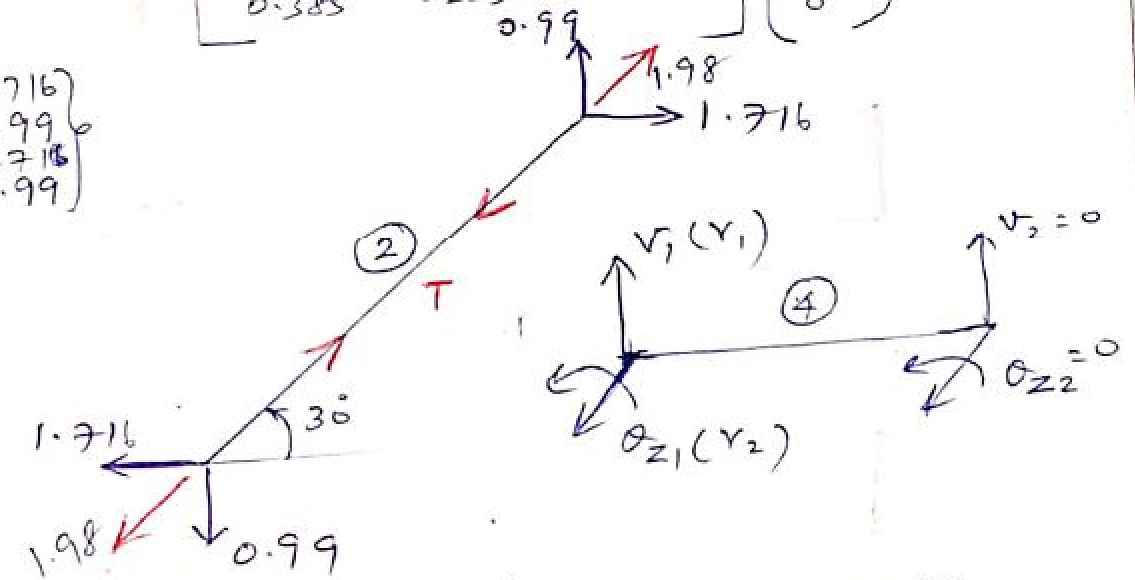
$$= \begin{Bmatrix} 1.38 \\ 2.377 \\ -1.38 \\ 1.78 \end{Bmatrix}$$



Element ② :

$$\{S\} = EI \begin{bmatrix} 0.667 & 0.385 & -0.667 & -0.385 \\ 0.385 & 0.223 & -0.385 & -0.223 \\ -0.667 & -0.385 & 0.667 & 0.385 \\ 0.385 & -0.223 & 0.385 & 0.223 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{-4.456}{EI} \\ 0 \\ 0 \end{Bmatrix} r_1$$

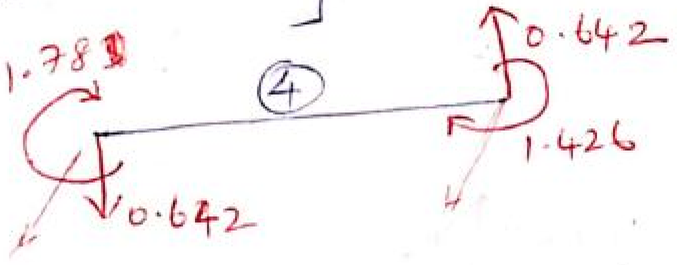
$$= \begin{Bmatrix} -1.716 \\ -0.99 \\ 1.716 \\ 0.99 \end{Bmatrix}$$

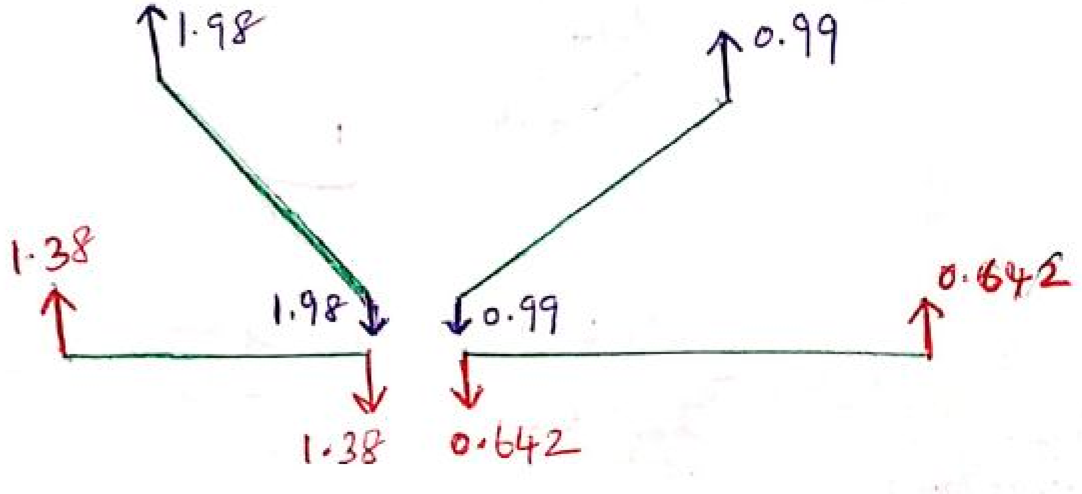


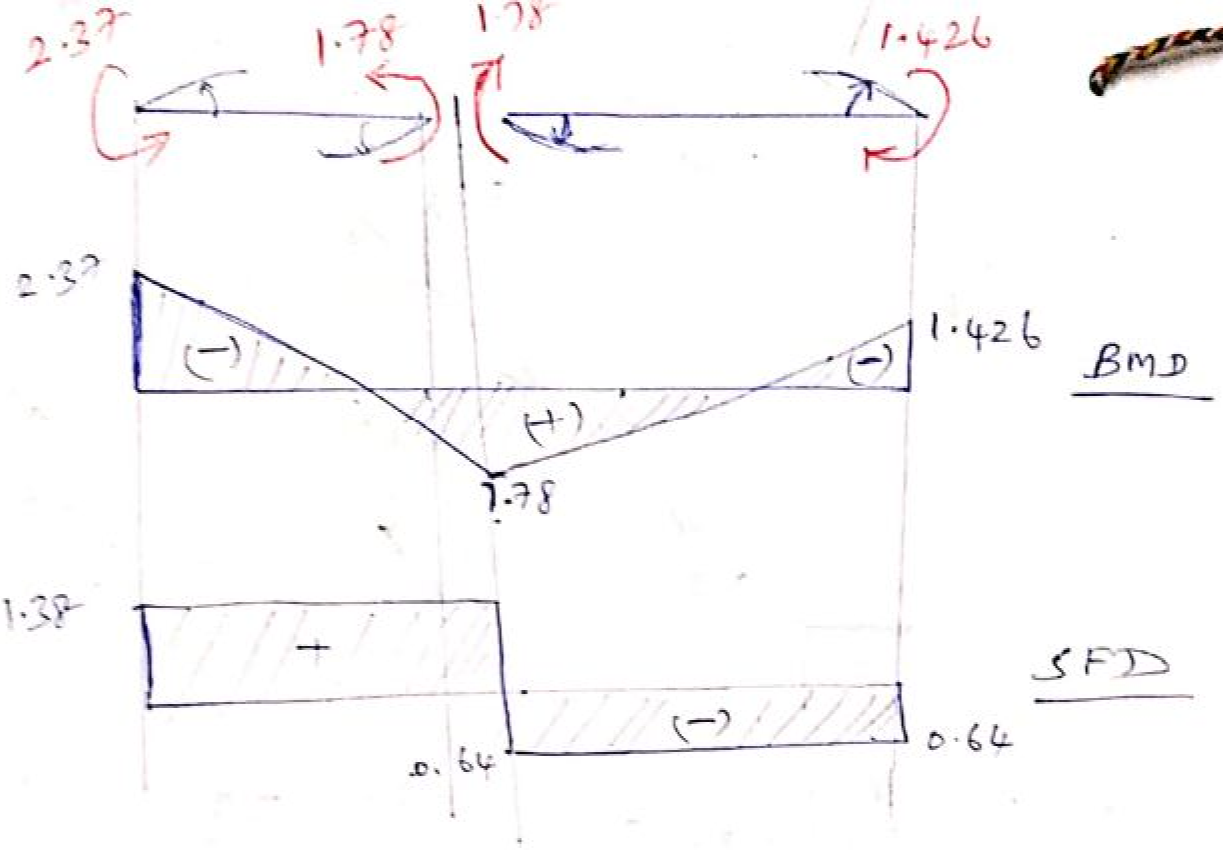
Element ④ :

$$\{S\} = \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.80 & -0.24 & 0.40 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.40 & -0.24 & 0.80 \end{bmatrix} \begin{Bmatrix} \frac{-4.456}{EI} \\ \frac{-0.892}{EI} \\ 0 \\ 0 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ \\ \end{matrix}$$

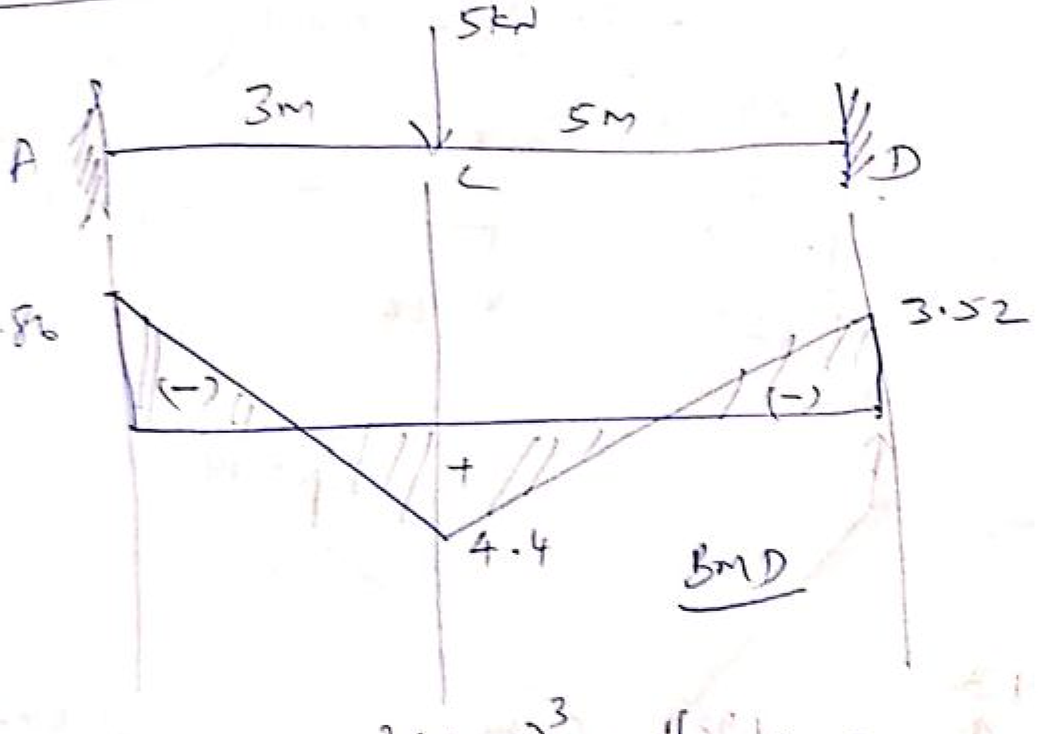
$$= \begin{Bmatrix} -0.642 \\ -1.783 \\ 0.642 \\ -1.426 \end{Bmatrix}$$







Fixed Beam:



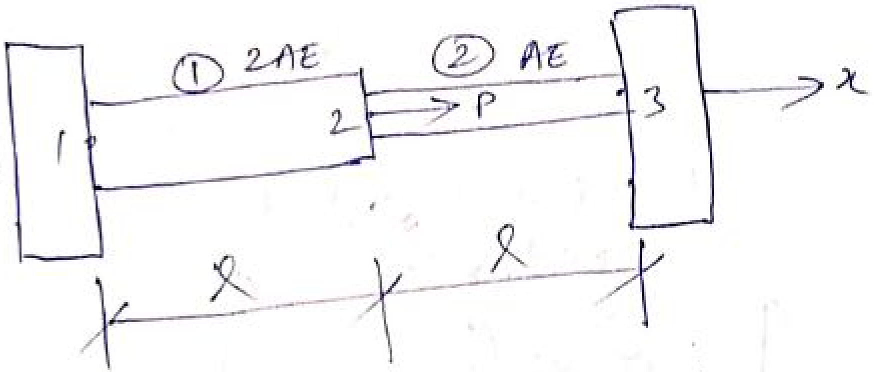
$$\delta_c = \frac{Wx^3(L-x)^3}{3L^3EI} \quad \Bigg| \quad \begin{array}{l} x = 3 \\ L = 8 \end{array}$$

$$= \frac{11}{EI}$$

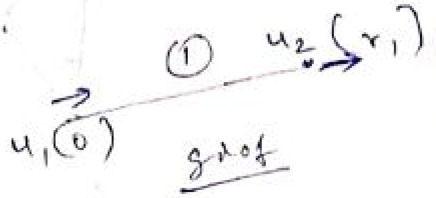
29/3/20
Saturday

UQ. (Nov. 2014) (72)

Find the stresses in two bar assembly which is loaded with force P and restrained at two two nodes as shown in figure.



bar → truss



$$[k_{1m}] = \begin{array}{cc|c} 0 & 1 & \text{g.dof} \\ \frac{2AE}{l} & -\frac{2AE}{l} & 0 \\ -\frac{2AE}{l} & \boxed{\frac{2AE}{l}} & 1 \end{array}$$

$$[K_{2m}] = \begin{array}{cc|c} 1 & 0 & \text{g.dof} \\ \boxed{\frac{AE}{l}} & -\frac{AE}{l} & 1 \\ -\frac{AE}{l} & \frac{AE}{l} & 0 \end{array}$$

$$[K] = \left[\frac{2AE}{l} + \frac{AE}{l}\right] = \left[\frac{3AE}{l}\right]_{1\times 1}$$

$$[K]\{r\} = \{Q\}$$

$$\left[\frac{3AE}{l}\right]\{r_1\} = \{P\} \Rightarrow \boxed{r_1 = \frac{Pl}{3AE}}$$

Element①:

$$\{\sigma\} = [C][B]\{d\}$$

σ_x

$$= E\,\frac{1}{l}\,[-1 \;\; 1]\begin{Bmatrix} 0 \\ Pl/3AE \end{Bmatrix}\begin{matrix} 0 \\ r_1 \end{matrix} = \frac{P}{3A} \;// \quad (T)$$

Element②

$$\sigma_x = E\,\frac{1}{l}\,[-1 \;\; 1]\begin{Bmatrix} \frac{Pl}{3AE} \\ 0 \end{Bmatrix} = \frac{-P}{3A} \quad (C)$$

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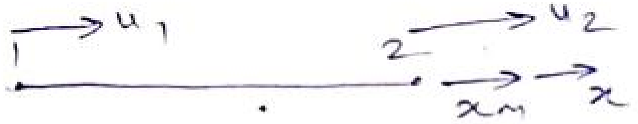
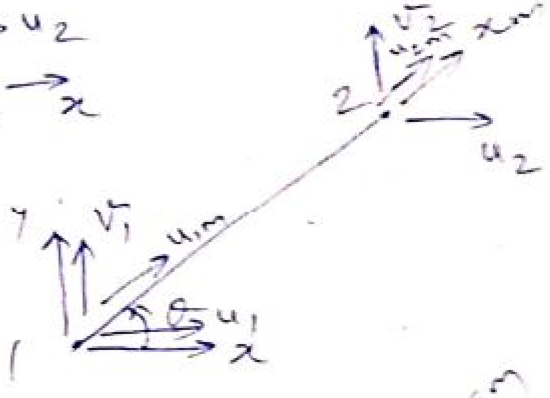


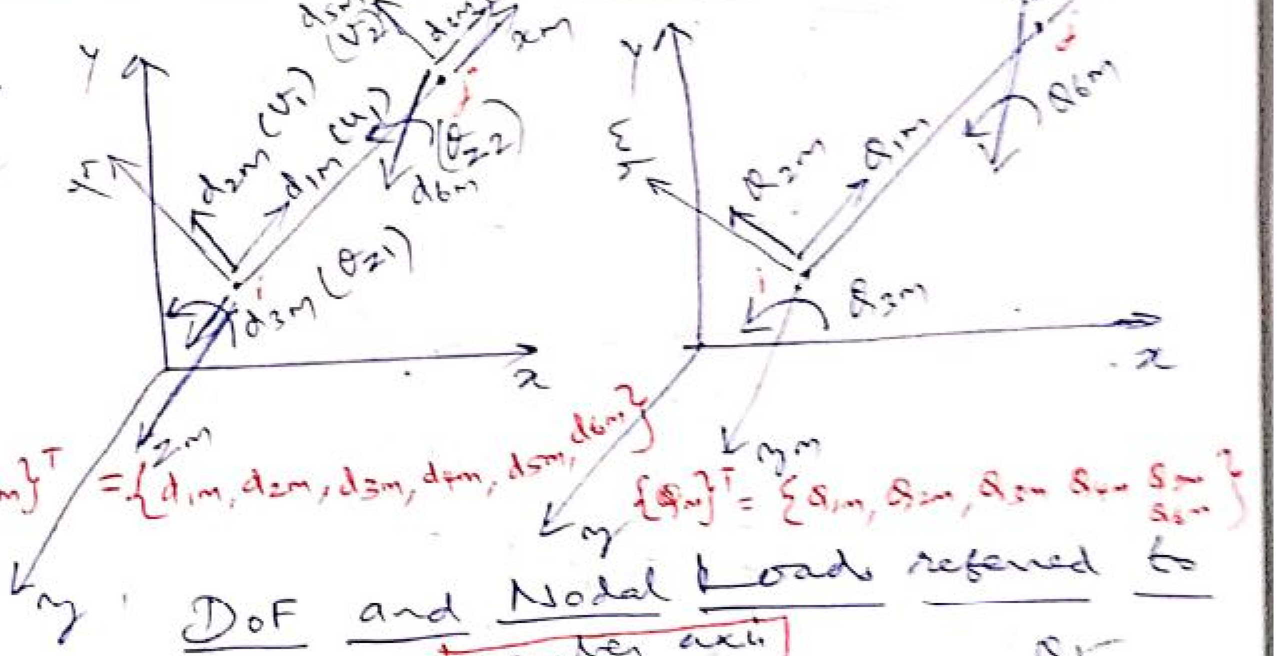
$[k]_{2\times2}$



$[k]_{4\times4}$

Rotation Transformation Matrix:

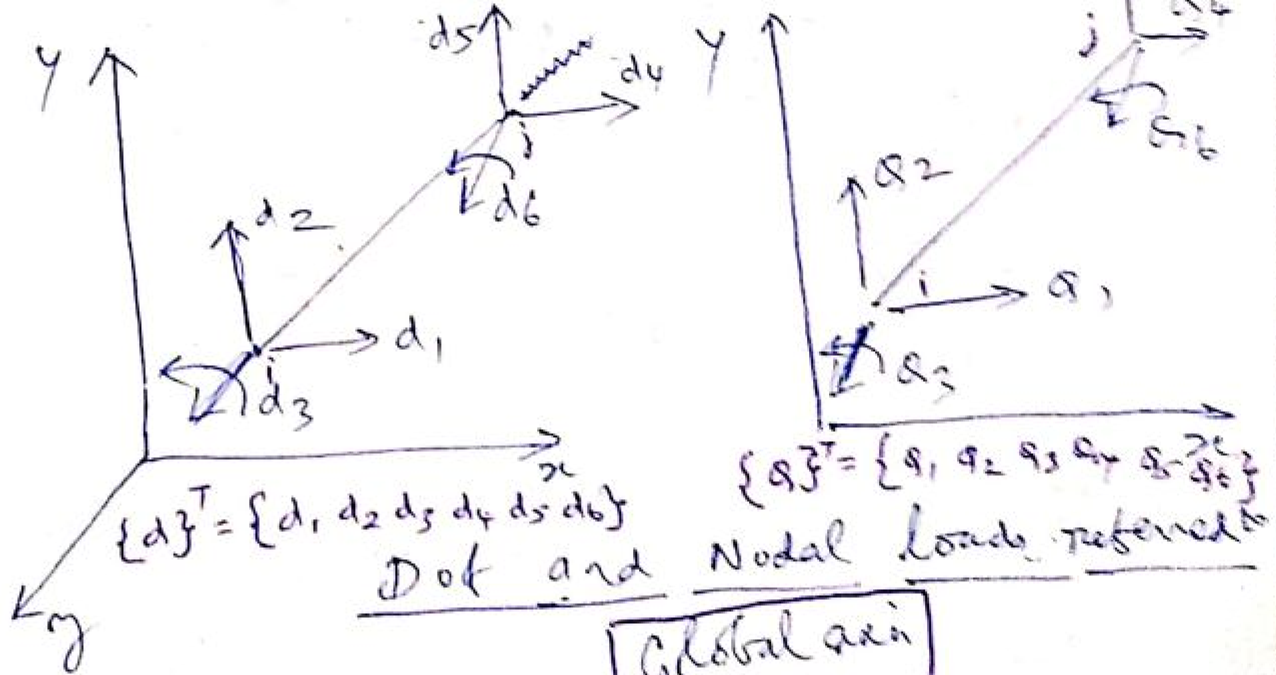
P222
CSK



$\{d_m\}^T = \{d_{1m}, d_{2m}, d_{3m}, d_{4m}, d_{5m}, d_{6m}\}$

$\{Q_m\}^T = \{Q_{1m}, Q_{2m}, Q_{3m}\ Q_{4m}\ Q_{5m}\ Q_{6m}\}$

DoF and Nodal Loads referred to member axis



$\{d\}^T = \{d_1\ d_2\ d_3\ d_4\ d_5\ d_6\}$

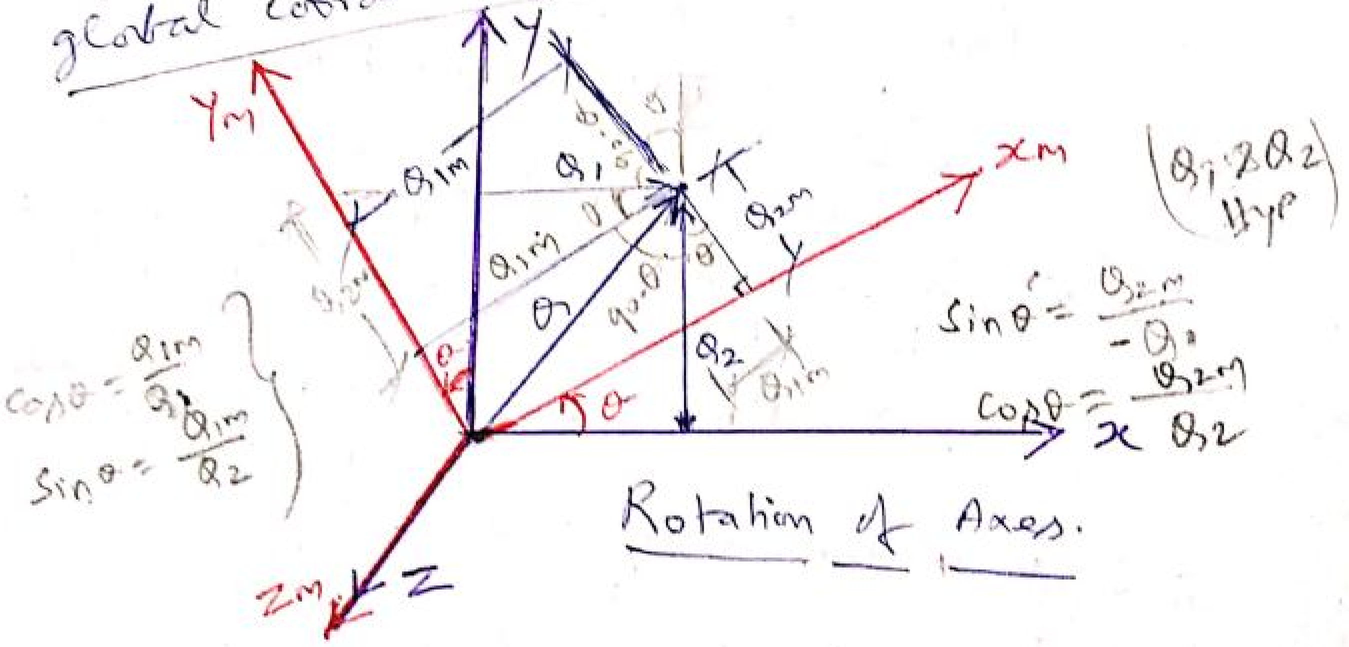
$\{Q\}^T = \{Q_1\ Q_2\ Q_3\ Q_4\ Q_5\ Q_6\}$

DoF and Nodal loads referred to Global axis

If the two dimensional beam element (73) is inclined as shown in figures, then the stiffness matrix will first be computed in local member axes system using the equation (8.62)

IIly, the nodal load vector $\{Q_m\}$ due to loads normal to the member will be negatives of the fixed end actions with respect to member axes.

For the assemblage of elements to the overall equations of equilibrium, transformation of these element properties is required to conform to the degrees of freedom and nodal forces expressed in the global coordinate system as shown figures.



Rotation of Axes.

Consider a local system of axes x_m and y_m in $x-y$ plane rotated through an angle θ as shown in figure. The local axis z_m is normal to the plane x_m-y_m and thus coincides with the global z axis.

Let the force Q has components Q_1 and Q Paralled to the global axes.

Resolving these force components into the axes x_m and y_m, we obtain,

$$Q_{1m} = Q_1 \cos\theta + Q_2 \sin\theta = Q_1 C_x + Q_2 C_y$$

$$Q_{2m} = -Q_1 \sin\theta + Q_2 \cos\theta = -Q_1 C_y + Q_2 C_x$$

where $C_x = \cos\theta$, $C_y = \sin\theta$

$$Q_{3m} = Q_3$$

In matrix form, [components of forces in local system (x_m, y_m, z_m) can be expressed in terms of their components in global system of axes (x, y, z)].

$$\begin{Bmatrix} Q_{1m} \\ Q_{2m} \\ Q_{3m} \end{Bmatrix} = \begin{bmatrix} C_x & C_y & 0 \\ -C_y & C_x & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix}$$

$$\{Q_m'\} = [T']\{Q'\}$$

where $\{Q_m'\}$ → member load at one end.

$[T']$ → Transformation matrix for one end.

$\{Q'\}$ → global load at one end.

It is also seen that the transpose of $[T']$ is equal to its inverse. $[T']^T = [T']^{-1}$

$$\therefore \{Q'\} = [T']^T \{Q_m'\}$$

force components in global system in term of their components in local system of axes.

Since, small displacements and forces can be treated as vectors, the above relationship can be used for displacement as well. So,

$$\{d_m'\} = [T']\{d'\}$$

$$\& \quad \{Q'\} = [T']^T \{d_m'\}$$

Where $\{d'_m\}$ and $\{d'\}$ refer to the displacements at one node with reference to the member and global axes respectively.

Considering the 2 Dimensional beam element the forces at the 2 nodes with reference to local and global axes can be related as

$$\begin{Bmatrix} Q_{1m} \\ Q_{2m} \\ Q_{3m} \\ Q_{4m} \\ Q_{5m} \\ Q_{6m} \end{Bmatrix}_{6\times1} = \left[\begin{array}{ccc|ccc} c_x & c_y & 0 & 0 & 0 & 0 \\ -c_y & c_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & c_x & c_y & 0 \\ 0 & 0 & 0 & -c_y & c_x & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]_{6\times6} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix}_{6\times1}$$

member load vector

Rotational Transformation matrix

Global load vector.

$$\{Q_m\} = [T]\{Q\}$$

$$\{Q\} = [T]^T\{Q_m\}$$

(or

$$\left\{\begin{array}{c} \{Q_m\}_1 \\ \hdashline \{Q_m\}_2 \end{array}\right\} = \left[\begin{array}{c:c} [T'] & [0] \\ \hdashline [0] & [T'] \end{array}\right] \left\{\begin{array}{c} \{Q_1\} \\ \hdashline \{Q_2\} \end{array}\right\}$$

i.e. $\{Q_m\} = [T]\{Q\}$

$|||^{ly}$ Nodal displacements,

$$\{d_m\} = [T]\{d\}$$

$$\{d\} = [T]^T\{d_m\}$$

Equilibrium equation:

$$[k_m]\{d_m\} = \{Q_m\}$$

$$[k_m][T]\{d\} = [T]\{Q\}$$

$$\underbrace{[T]^T[k_m][T]}\{d\} = \{Q\}$$

$$[K]\{d\} = \{Q\}$$

$$\therefore \boxed{[K] = [T]^T[k_m][T]}$$

[K] is stiffness matrix of element in global axes system and is obtained by transforming [k_m], the stiffness matrix in the member axes system.

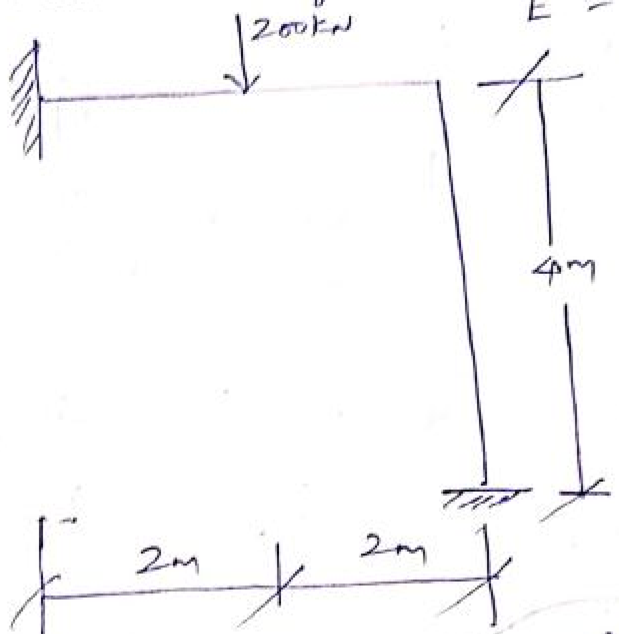
31/3/20 Monday

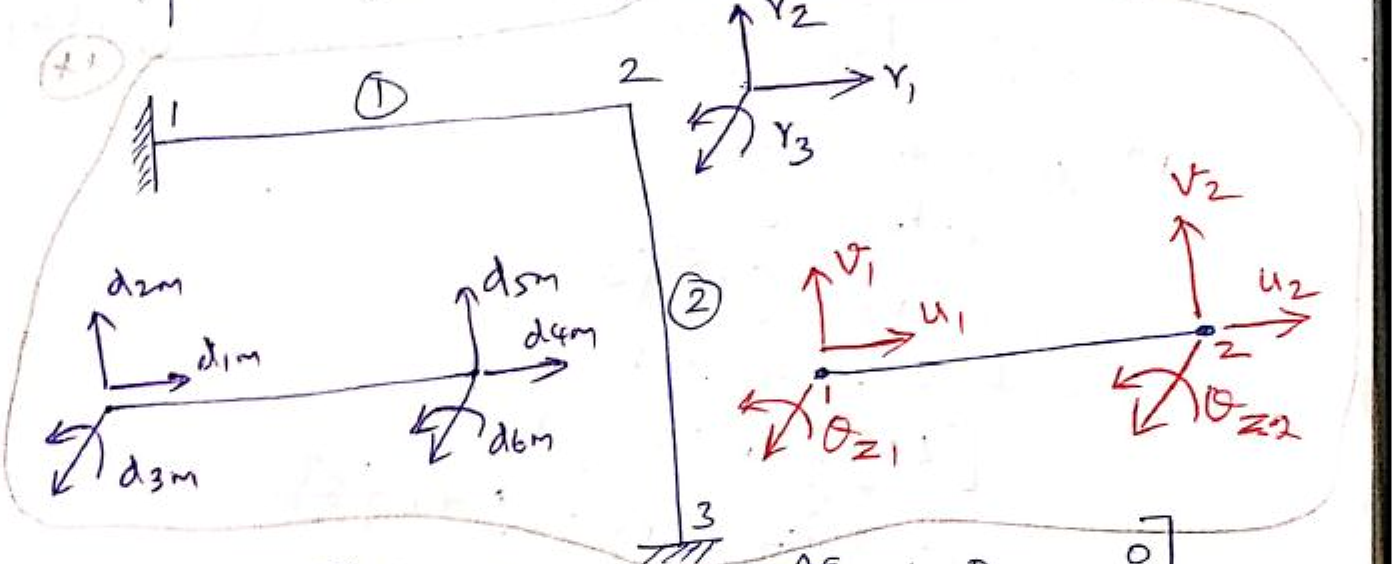
Analyse the plane frame shown in figure using 2D beam element. Draw the bending moment diagram.

$A = 0.03\ m^2$

$E = 2 \times 10^7\ kN/m^2$

$I_Z = 12 \times 10^{-5}\ m^4$





$$[k_m] = \begin{bmatrix} \frac{AE}{l} & 0 & 0 & -\frac{AE}{l} & 0 & 0 \\ 0 & \frac{12EI}{l^3} & \frac{6EI}{l^2} & 0 & -\frac{12EI}{l^3} & \frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{4EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{2EI}{l} \\ -\frac{AE}{l} & 0 & 0 & \frac{AE}{l} & 0 & 0 \\ 0 & -\frac{12EI}{l^3} & -\frac{6EI}{l^2} & 0 & \frac{12EI}{l^3} & -\frac{6EI}{l^2} \\ 0 & \frac{6EI}{l^2} & \frac{2EI}{l} & 0 & -\frac{6EI}{l^2} & \frac{4EI}{l} \end{bmatrix}$$

Pb2

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75

$$\frac{AE}{l} = \frac{0.03 \times 2\times10^{7}}{4} = 15\times10^{4}$$

$$\frac{12EI}{l} = \frac{12\times2\times10^{7}\times12\times10^{-5}}{4^{3}} = 450$$

$$\frac{6EI}{l^2} = \frac{6\times2\times10^{7}\times12\times10^{-5}}{4^{2}} = 900$$

$$\frac{4EI}{l} = \frac{4\times2\times10^{7}\times12\times10^{-5}}{4} = 2400$$

$$\frac{2EI}{l} = \frac{2\times2\times10^{7}\times12\times10^{-5}}{4} = 1200$$

Element ①

Connectivity 1-2

Since, the member is oriented in the global direction, no transformation is required. ($\theta = 0$)

Rotation matrix is identity matrix.

$$[K_1] = [K_m] = \begin{array}{c|cccccc} & 0 & 0 & 0 & 1 & 2 & 3 \\ \hline 0 & 15\times10^4 & 0 & 0 & -15\times10^4 & 0 & 0 \\ 0 & 0 & 450 & 900 & 0 & -450 & 900 \\ 0 & 0 & 900 & 2400 & 0 & -900 & 1200 \\ 1 & -15\times10^4 & 0 & 0 & 15\times10^4 & 0 & 0 \\ 2 & 0 & -450 & -900 & 0 & 450 & -900 \\ 3 & 0 & 900 & 1200 & 0 & -900 & 2400 \end{array}$$

$\leftarrow$ gdof $\downarrow$

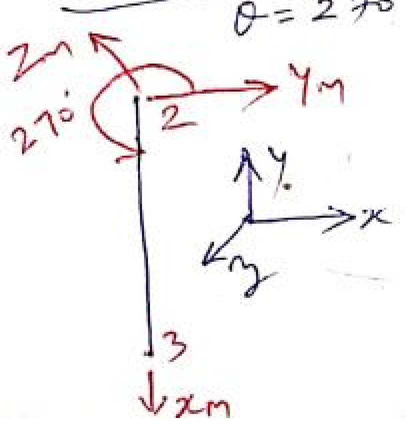
$[K_1]$ member stiffness matrix in global

$[K_m]$ member stiffness matrix in Element

Element ②

$\theta = 270°$

$C_x = \cos 270° = 0$, $C_y = \sin 270° = -1$



$$T = \left[\begin{array}{ccc|ccc} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

(P74)

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31/1

$$[k_2] = [T]^T [k_m] [T]$$

$$[k_2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 15\times10^4 & 0 & 0 & -15\times10^4 & 0 & 0 \\ 0 & 450 & 900 & 0 & -450 & 900 \\ 0 & 900 & 2400 & 0 & -900 & 1200 \\ -15\times10^4 & 0 & 0 & 15\times10^4 & 0 & 0 \\ 0 & -450 & -900 & 0 & 450 & -900 \\ 0 & 900 & 1200 & 0 & -900 & 2400 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

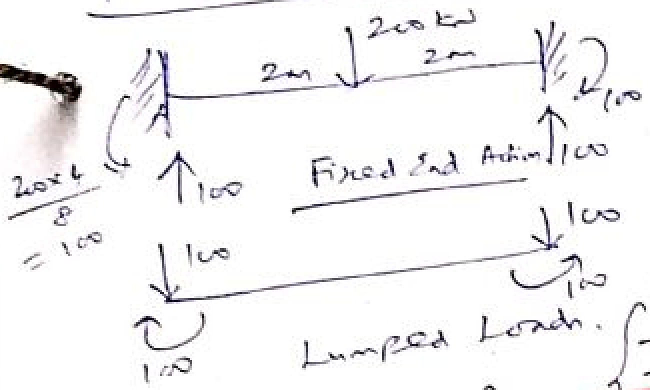
$$[k_2] = \begin{bmatrix} 0 & 450 & 900 & 0 & -450 & 900 \\ -15\times10^4 & 0 & 0 & 15\times10^4 & 0 & 0 \\ 0 & 900 & 2400 & 0 & -900 & 1200 \\ 0 & -450 & -900 & 0 & 450 & -900 \\ +15\times10^4 & 0 & 0 & 15\times10^4 & 0 & 0 \\ 0 & 900 & 1200 & 0 & -900 & 2400 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

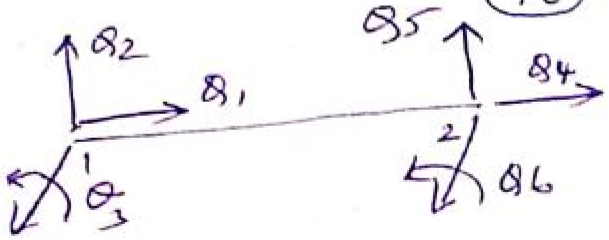
$$\begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 0 & 0 & 0 \leftarrow \text{gdof} \downarrow \end{array} & \\ = \begin{bmatrix} 450 & 0 & 900 & -450 & 0 & 900 \\ 0 & 15\times10^4 & 0 & 0 & -15\times10^4 & 0 \\ 900 & 0 & 2400 & -900 & 0 & 1200 \\ -450 & 0 & -900 & 450 & 0 & -900 \\ 0 & -15\times10^4 & 0 & 0 & -15\times10^4 & 0 \\ 900 & 0 & 1200 & -900 & 0 & 2400 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{array} \end{array}$$

Figures (x x) **Assembling:**

$$K = \begin{array}{c c} \begin{bmatrix} \begin{matrix} 15\times10^4 \\ 450 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 900 \end{matrix} \\ \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 450 \\ 15\times10^4 \end{matrix} & \begin{matrix} -900 \\ 0 \end{matrix} \\ \begin{matrix} 0 \\ 900 \end{matrix} & \begin{matrix} -900 \\ 0 \end{matrix} & \begin{matrix} 2400 \\ 2400 \end{matrix} \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} & \end{array} = \begin{bmatrix} 15.045\times10^4 & 0 & 900 \\ 0 & 15.045\times10^4 & -900 \\ 900 & -900 & 4800 \end{bmatrix}$$

Nodal Load vector





Lumped Load.

$$\{Q_1\} = \{Q_{1m}\} = \begin{Bmatrix} 0 \\ -100 \\ -100 \\ 0 \\ -100 \\ 100 \end{Bmatrix} \begin{array}{c} \text{g.dof} \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$

$$\{Q_2\} = [T]^T \{Q_{2m}\} = \{0\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} \text{g.dof} \\ 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{array}$$

Equilibrium Equation:

$$[K]\{r\} = \{Q\}$$

$$\begin{bmatrix} 15.045 \times 10^4 & 0 & 900 \\ 0 & 15.045 \times 10^4 & -900 \\ 900 & -900 & 4800 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -100 \\ 100 \end{Bmatrix}$$

P69

$$|A| = 150450(150450 \times 4800 - (-900)^2) - 0 + 900(0 - 900 \times 150450)$$

$$= 1.08405243 \times 10^{14}$$

$$C = \begin{bmatrix} + \begin{vmatrix} 150450 & -900 \\ -900 & 4800 \end{vmatrix} & - \begin{vmatrix} 0 & -900 \\ 900 & 4800 \end{vmatrix} & + \begin{vmatrix} 0 & 150450 \\ 900 & -900 \end{vmatrix} \\ - \begin{vmatrix} 0 & 900 \\ -900 & 4800 \end{vmatrix} & + \begin{vmatrix} 150450 & 900 \\ 900 & 4800 \end{vmatrix} & - \begin{vmatrix} 150450 & \overset{0}{-900} \\ 900 & \cancel{150450} \end{vmatrix} \\ + \begin{vmatrix} 0 & 900 \\ 150450 & -900 \end{vmatrix} & - \begin{vmatrix} 150450 & 900 \\ \overset{0}{\cancel{900}} & -900 \end{vmatrix} & + \begin{vmatrix} 150450 & 0 \\ 0 & 150450 \end{vmatrix} \end{bmatrix}$$

$$C = \begin{bmatrix} 721350000 & -810000 & -135405000 \\ -810000 & 721350000 & +135405000 \\ -135405000 & +135405000 & 22,635,202,500 \end{bmatrix}$$

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$$C^{T} = \begin{bmatrix} 721350000 & -810000 & -135405000 \\ -810000 & 721350000 & 135405000 \\ -135405000 & 135405000 & 22635202500 \end{bmatrix}$$

$$[A]^{-1} = \frac{C^{T}}{|A|} = \begin{bmatrix} 0.0000066542 & -0.0000000075 & -0.000001249 \\ -0.0000000075 & 0.0000066542 & 0.000001249 \\ -0.000001249 & 0.000001249 & 0.0002088017 \end{bmatrix}$$

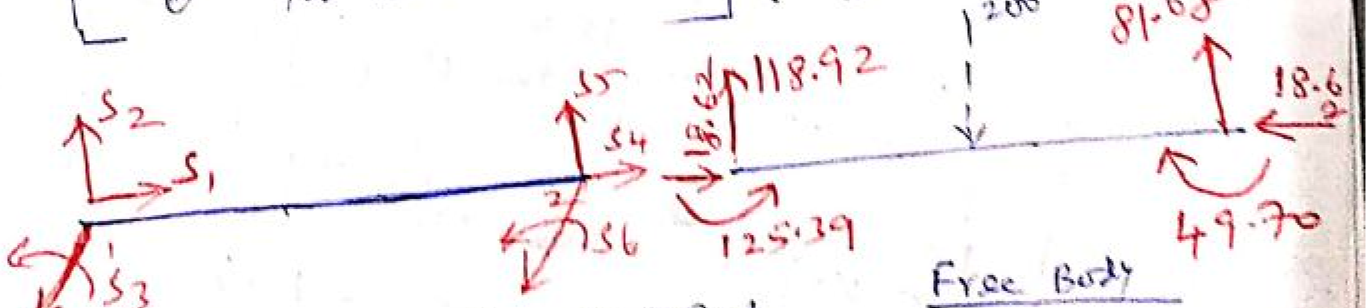
$$\begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 0 \\ -100 \\ 100 \end{Bmatrix} = \begin{Bmatrix} -0.00012415 \\ -0.00054052 \\ 0.02075527 \end{Bmatrix} \begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}$$

Member end action : $\{S\} = [k_m]\{d_m\} + \{S_0\}$

$\{d_m\} = [T]\{d\}$

Member ①: $\{d_m\} = \{d\}$

$$\{S_1\} = \begin{bmatrix} 150000 & 0 & 0 & -150000 & 0 & 0 \\ 0 & 450 & 900 & 0 & -450 & 900 \\ 0 & 900 & 2400 & 0 & -900 & 1200 \\ -150000 & 0 & 0 & 150000 & 0 & 0 \\ 0 & -450 & -900 & 0 & 450 & -900 \\ 0 & 900 & 1200 & 0 & -900 & 2400 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ r_1 \\ r_2 \\ r_3 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 100 \\ 100 \\ 0 \\ 100 \\ -100 \end{Bmatrix} = \begin{Bmatrix} 18.62 \\ 118.92 \\ 125.39 \\ -18.62 \\ 81.08 \\ -49.70 \end{Bmatrix}$$

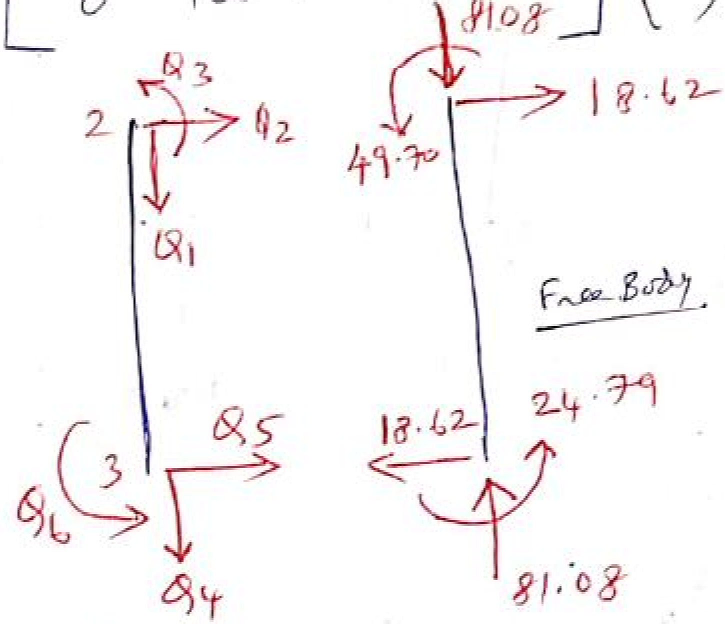


Free Body

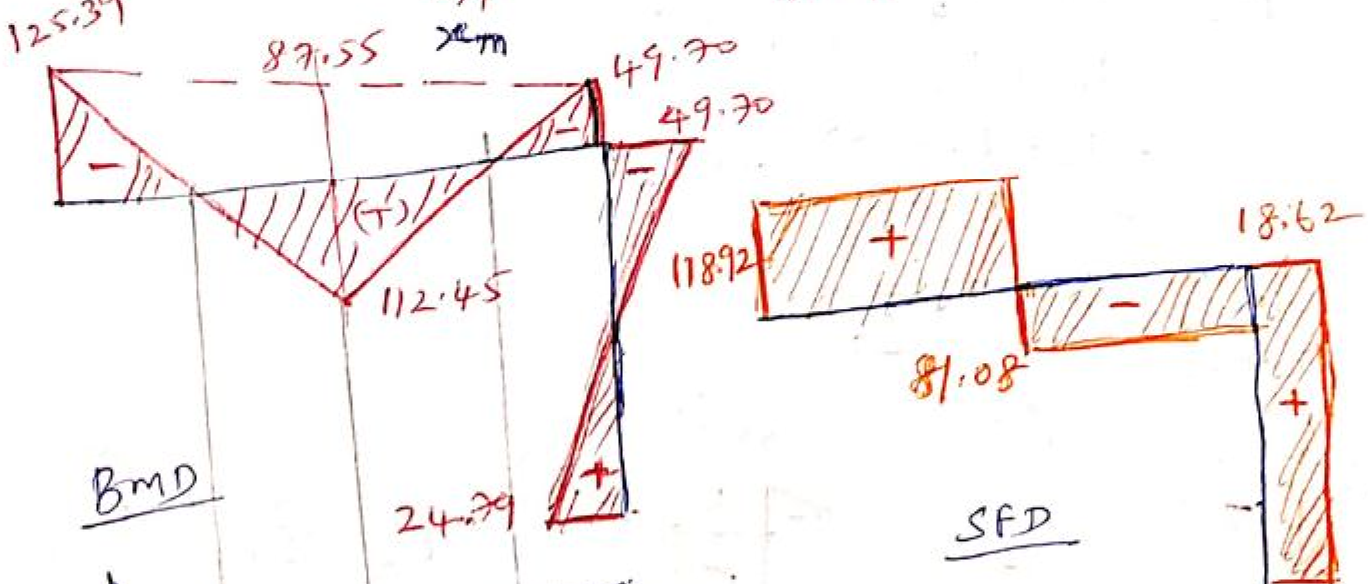
Member ②: $\{d_m\} = [T]\{d\}$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.00054 \\ -0.00012415 \\ 0.0207552 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{S_2\} = \begin{bmatrix} 150000 & 0 & 0 & -150000 & 0 & 0 \\ 0 & 450 & 900 & 0 & -450 & 900 \\ 0 & 900 & 2400 & 0 & -900 & 1200 \\ -150000 & 0 & 0 & 150000 & 0 & 0 \\ 0 & -450 & -900 & 0 & 450 & -900 \\ 0 & 900 & 1200 & 0 & -900 & 2400 \end{bmatrix} \begin{Bmatrix} -Y_2 \\ Y_1 \\ Y_3 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 81.08 \\ 18.62 \\ 49.70 \\ -81.08 \\ -18.62 \\ 24.79 \end{Bmatrix}$$

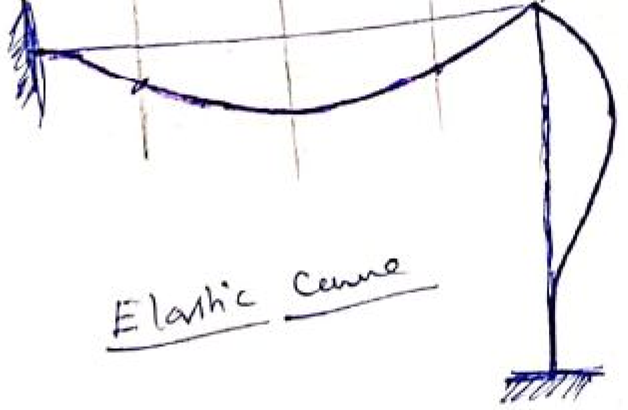


Free Body



BMD

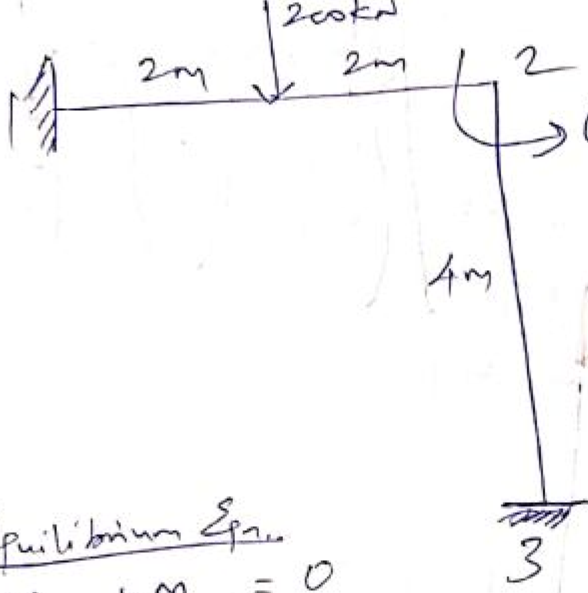
SFD



Elastic Curve

1/4/20
Tuesday.

Slope deflection method.



$\theta_2 = \gamma = \gamma_3$

$$\curvearrowleft M_{1-2} = 100$$

$$\curvearrowleft M^f_{21} = \frac{200 \times 4}{8} = -100 \text{ kNm}$$

Slope deflection Equation

$$M_{2-1} = M^f_{21} + \frac{2EI}{l}(2\theta_2 + \cancel{\theta_1}^{0})$$

$$= -100 + \frac{\cancel{2}EI}{\cancel{4}}(\cancel{2}\theta_2)$$

$$= -100 + EI\theta_2 \quad ①$$

$$M_{2-3} = 0 + \frac{2EI}{4}(2\theta_2 + \cancel{\theta_3}^{0})$$

$$= EI\theta_2 \quad ②$$

~~[illegible]~~

Equilibrium Eqn.

$$M_{2-1} + M_{2-3} = 0$$

$$-100 + EI\theta_2 + EI\theta_2 = 0$$

$$EI\theta_2 = 100/2 = 50$$

$$\theta_2 = 50/EI.$$

$$M_{1-2} = 100 + \frac{2EI}{4}(2\cancel{\theta_1}^{0} + \theta_2)$$

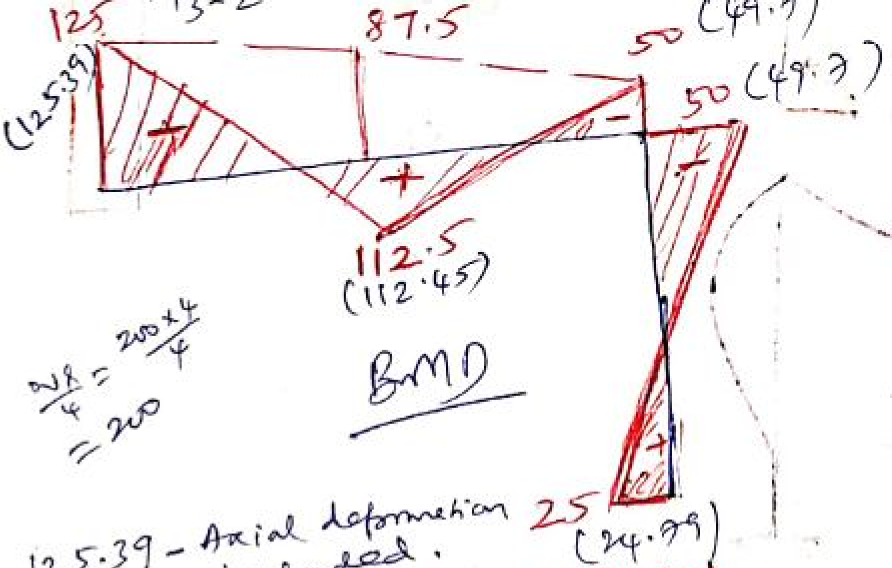
$$= 100 + \frac{2EI}{4}(50/E\cancel{I})$$

$$= 125 \text{ kNm} \curvearrowleft$$

$$M_{2-1} = -100 + EI \times \frac{50}{EI} = -50 \text{ kNm} \curvearrowright$$

$$M_{2-3} = 50 \text{ kNm} \curvearrowleft$$

$$M_{3-2} = 25 \text{ kNm} \curvearrowleft$$



$$\theta_2 = \frac{50}{EI} = \frac{50}{2 \times 10^7 \times 12 \times 10^{-5}}$$

$$= 0.0208333333$$

(0.02075527)

$$\frac{wl}{4} = \frac{200 \times 4}{4} = 200$$

BMD

125.39 - Axial deformation included.

125 - Axial deformation neglected.